Polar, cylindrical and spherical coordinates

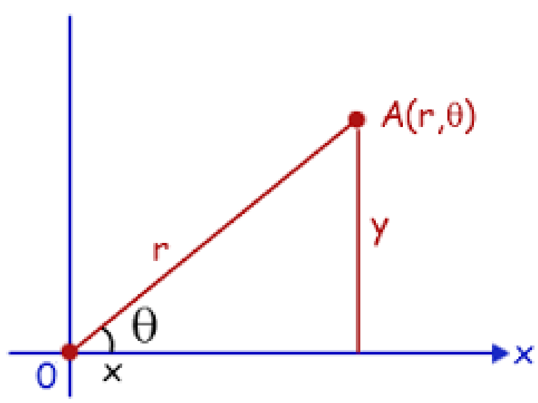
Math 251

October 17, 2015

1 Polar coordinates

Consider a point P on the plane with given x, y axis. Denote the origin $(0, 0)$ as O. To exactly describe P, we can prescribe its (x, y) coordinates. That is we prescribe how far P is away from O along the x-axis and along the y-axis.

Alternatively, we can prescribe the distance $r = |OP|$ and the angle θ between OP and the positive $x - axis$. You should verify that this also uniquely describe P with the restriction that $r > 0$.



Indeed, we have

$$\begin{aligned} r^2 &= x^2 + y^2 \\ \theta &= \tan^{-1}\left(\frac{y}{x}\right), \end{aligned}$$

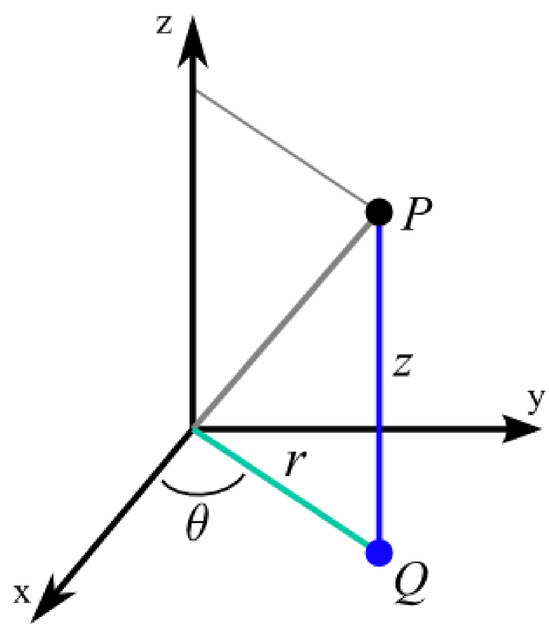
and

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta). \end{aligned}$$

That is there is a one to one correspondence between the (x, y) and (R, θ) coordinate system. We refer to (R, θ) as the polar coordinate system.

2 Cylindrical coordinates

Now consider P with given (x, y, z) coordinate. If we transform (x, y) to the corresponding polar coordinate, then we can describe P as (r, θ, z). This is referred to as the cylindrical coordinate system.



3 Spherical coordinates

Again consider P with given (x, y, z) coordinate. Let P' be the projection of P on the xy-plane. That is $P' = (x, y, 0)$. We can alternatively describe P with

a) ρ: the distance $|OP|$

b) θ: the angle between OP' and the positive x axis

c) ϕ: the angle between OP and the positive z axis.

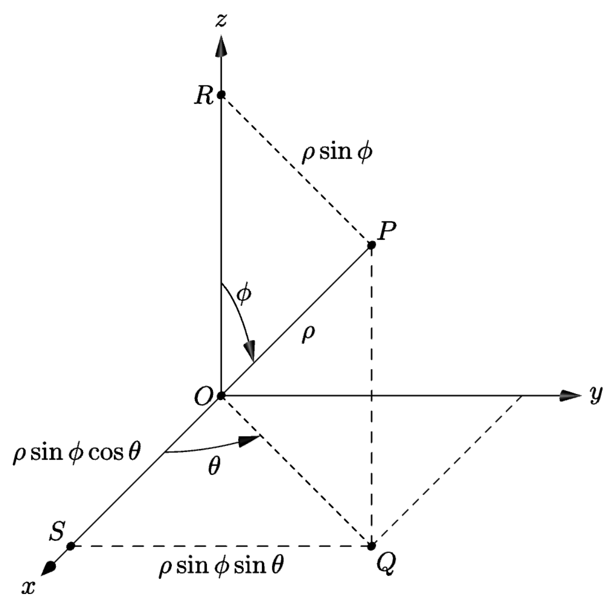
The coordinate (ρ, θ, ϕ) and is referred to as the spherical coordinate. We make the restriction that $\rho > 0, 0 \leq \phi \leq \pi$. The spherical coordinate also uniquely describes P

as followed:

$$\begin{aligned}\rho^2 &= x^2+y^2+z^2\\ \theta &= \tan^{-1}\left(\frac{y}{x}\right)\\ \phi &= \cos^{-1}\left(\frac{z}{\rho}\right).\end{aligned}$$

and

$$\begin{aligned}x &= \rho\sin\phi\cos\theta\\ y &= \rho\sin\phi\sin\theta\\ z &= \rho\cos\phi.\end{aligned}$$



4 Change of variables in polar coordinates

Let $\mathcal{R}$ be the circle with radius 1 around $(0,0)$ and consider the double integral

$$\iint_{\mathcal{R}} xy^2\,dx\,dy. \tag{1}$$

Because the region is a circle, intuitively it is more convenient to use polar coordinate to integrate. The limit of the double integral is clear: r goes from 0 to 1, θ goes from 0 to 2π. The substitution is also clear: $x \to r\cos\theta, y \to r\sin\theta$. What about the differential $dx\,dy$? It turns out the correct substition for $dx\,dy$ is $r\,dr\,d\theta$. We will give an explanation for this substitution in the section on general change of variables. For now the intuition can be captured via the following picture: If we approximate

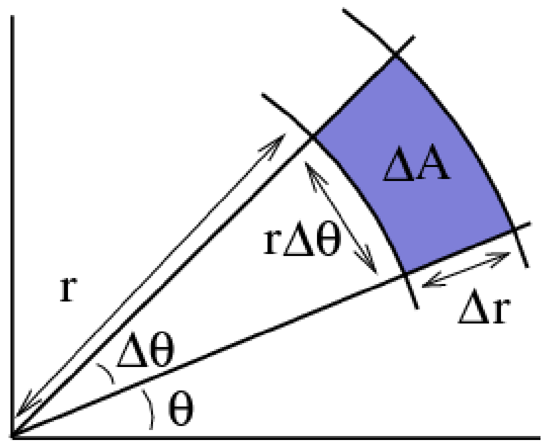


Figure 4.1: The differential of the area in polar coordinate

ΔA as a rectangle with width Δr and length $r\Delta\theta$ (which is the arc length of the arc with angle $\Delta\theta$ and radius r) then $\Delta A \approx (\Delta r)(r\Delta\theta)$.

Thus the integral (1) becomes

$$\int_0^{2\pi}\int_0^1 (r\cos\theta)(r\sin\theta)^2\, r\,dr\,d\theta.$$

In general we have

$$\iint_{\mathcal{R}} f(x,y)dxdy = \iint_{\mathcal{R}} f(r\cos\theta, r\sin\theta)\, r\,dr\,d\theta,$$

where on the RHS we need to convert the region $\mathcal{R}$ to its appropriate expression in polar coordinate.

5 Change of variables in cylindrical coordinates

Consider the triple integral

$$\iiint_{\mathcal{R}} f(x,y,z)\,dx\,dy\,dz.$$

Following exactly the same argument as above we have

$$\iiint_{\mathcal{R}} f(x,y,z)\,dx\,dy\,dz = \iint_{\mathcal{R}'}\left(\int_{g_1(r,\theta)}^{g_2(r,\theta)} f(r\cos\theta, r\sin\theta, z)dz\right)\, r\,dr\,d\theta,$$

where R' is a region on the plane and g_1, g_2 are the limits of z in the original region $\mathcal{R}$ expressed in polar coordinate.

6 Change of variables in spherical coordinates

Consider the triple integral

$$\iiint_{\mathcal{R}} f(x, y, z)\, dx\, dy\, dz.$$

where $\mathcal{R}$ is the unit sphere centered at the origin. Then it is natural to use spherical coordinate here where ρ goes from 0 to 1, θ goes from 0 to 2π and ϕ goes from 0 to π. The substitution of x, y, z in terms of ρ, θ, ϕ is also straight forward. Again the question is how do we substitute $dx\, dy\, dz$? The correct substition is $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. The following picture gives the intuition and we will give a mathematical explanation later on: Thus the original triple integral becomes

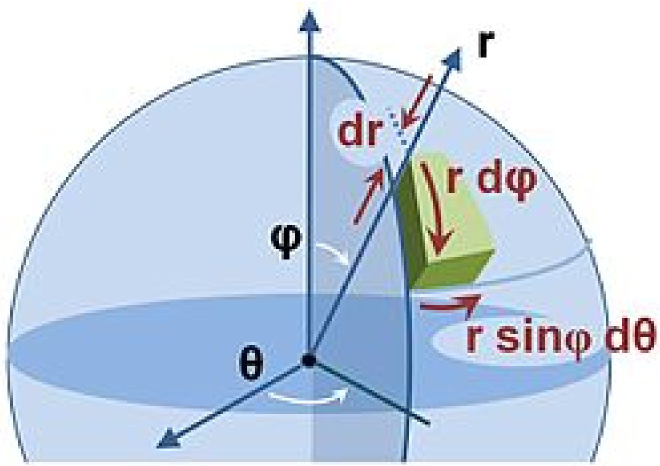


Figure 6.1: The differential of the volume in spherical cooridnates

$$\iiint_{\mathcal{R}} f(x, y, z)\, dx\, dy\, dz = \iiint_{\mathcal{R}} f(\rho \sin\phi \cos\theta, \rho \sin\phi \sin\theta, \rho \cos\phi)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta,$$

where the region $\mathcal{R}$ on the RHS needs to be expressed in appropriate spherical coordinate.