Integration in multi-variables

Math 251

October 17, 2015

1 Definition of the double integral

Let $f(x,y) : \mathbb{R}^2 \to \mathbb{R}$. Suppose for the moment that $f \geq 0$ and consider f on the rectangle $[a,b]\times[c,d]$. This generates a solid with base being the rectangle $[a,b]\times[c,d]$ and height $f(x,y), (x,y) \in [a,b]\times[c,d]$. We want to find the volume V of this solid.

To do this, we partition $[a,b]$ into M subintervals:

$$a = x_0 < x_1 < x_2 < \cdots < x_M = b,$$

and $[c,d]$ into N subintervals:

$$c = y_0 < y_1 < y_2 < \cdots < y_N = d.$$

This results in partitioning the rectangle $[a,b]\times[c,d]$ into $M\times N$ sub-rectangles. The volume of f over each sub-rectangle $[x_i, x_{i+1}]\times[y_j, y_{j+1}]$ can be approximated by

$$V_{ij} \approx f(x_i, y_j)(x_{i+1} - x_i)(y_{j+1} - y_j).$$

Thus the original volumn V can be approximated by

$$\begin{aligned} V &\approx \sum_{i,j} V_{ij} \\ &= \sum_{i,j} f(x_i, y_j)(x_{i+1} - x_i)(y_{j+1} - y_j). \end{aligned} \tag{1}$$

Now if we let the size of the partitions go to zero (on both the $[a,b]$ and $[c,d]$ intervals) then intuitively the double sum above converge to a number. We define this to be the double integral of f over $[a,b]\times[c,d]$:

$$\iint_{[a,b]\times[c,d]} f(x,y)dxdy := \lim_{\delta\to 0} \sum_{i,j} f(x_i, y_j)(x_{i+1} - x_i)(y_{j+1} - y_j),$$

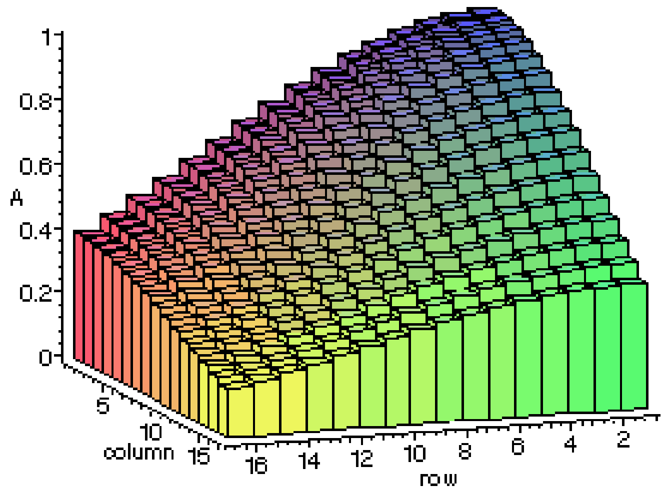

Figure 1.1: Riemann approximation of a double integral

where δ is the maximum length of the subintervals on $[a,b]$ and $[c,d]$.

The practical question for us is how to compute the double integral. To answer this we need to first discuss the iterated integral.

2 The iterated integrals

Consider $f(x,y): \mathbb{R}^2 \to \mathbb{R}, (x,y) \in [a,b] \times [c,d]$ as above. If we fix $y = y_0$ then $f(x,y)$ is a function in x. Thus we can define

$$\int_a^b f(x,y_0)dx.$$

Since y_0 is just a generic point in $[c,d]$ we can consider in general

$$F(y) = \int_a^b f(x,y_0)dx,$$

which is clearly a function of y in $[c,d]$. Thus we can again integrate $F(y)$ over $[c,d]$ to obtain

$$\int_c^d F(y)dy = \int_c^d \left(\int_a^b f(x,y)dx \right) dy.$$

Remark: This iteration of integration, first over y then over x can be interpreted as another approach to find the volume of f over $[a,b]\times[c,d]$.

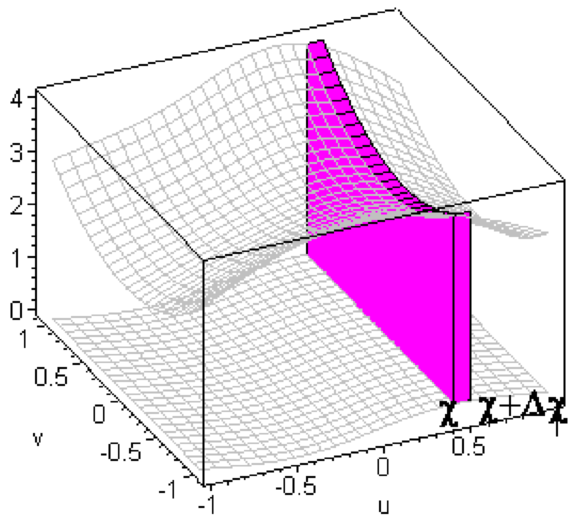

Figure 2.1: Iterated integral over x then over y

Similarly, if we fix x first, integrate in y and then in x we will obtain the iterated integral

$$\int_a^b \left(\int_c^d f(x,y)dy\right)dx.$$

In particular, if $f(x,y)$ has the form $f(x,y)=g(x)h(y)$ then we can easily evaluate the iterated integrals:

$$\begin{aligned}\int_c^d \left(\int_a^b f(x,y)dx\right)dy &= \left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right)\\ \int_a^b \left(\int_c^d f(x,y)dy\right)dx &= \left(\int_c^d h(y)dy\right)\left(\int_a^b g(x)dx\right).\end{aligned}$$

Intuitively, say fix x and if $f(x,y)=g(x)h(y)$ then the vertical trace of $f(x,y)$ (as a function of y) is just a constant times $h(y)$. So as x runs from a to b the vertical traces in y of $f(x,y)$ are all just mutiples of $h(y)$. That is the graph of $f(x,y)$ is "simple" in a certain way. Below are two graphs that can help you visualize this particular form of f.

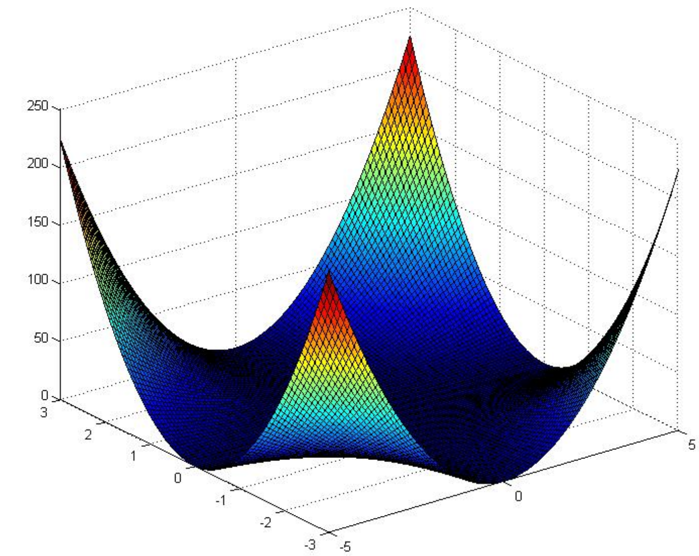

Figure 2.2: $f(x, y) = x^2y^2$ over $[-3, 3] \times [-5, 5]$

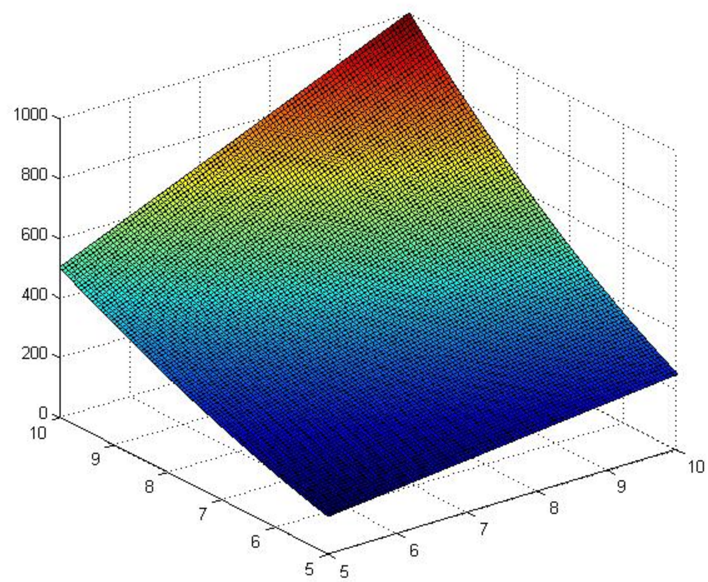

Figure 2.3: $f(x, y) = x^2y$ over $[5, 10] \times [5, 10]$

Note that in this case, the double sum (1) is

$$\begin{aligned}\sum_{i,j} f(x_i, y_j)(x_{i+1} - x_i)(y_{j+1} - y_j) &= \sum_{i,j} g(x_i) f(y_j)(x_{i+1} - x_i)(y_{j+1} - y_j) \\ &= \Big(\sum_i g(x_i)(x_{i+1} - x_i)\Big)\Big(\sum_j f(y_j)(y_{j+1} - y_j)\Big),\end{aligned}$$

which converges to

$$\left(\int_a^b g(x)dx\right)\left(\int_c^d h(y)dy\right).$$

Thus we suspect that

$$\begin{aligned}\iint_{[a,b]\times[c,d]} f(x,y)dxdy &= \int_c^d \left(\int_a^b f(x,y)dx\right) dy \qquad (2)\\ &= \int_a^b \left(\int_c^d f(x,y)dy\right) dx. \qquad (3)\end{aligned}$$

The fact that (2) is true in general (not just for $f(x,y) = g(x)h(y)$) is referred to as the Fubini's theorem. More precisely we have

Theorem 2.1. *(Fubini)*

Let $f(x,y)$ be such that

$$\iint_{[a,b]\times[c,d]} |f(x,y)|dxdy$$

exists (that is the double sum (1) converges for $|f(x,y)|$) then

$$\begin{aligned}\iint_{[a,b]\times[c,d]} f(x,y)dxdy &= \int_c^d \left(\int_a^b f(x,y)dx\right) dy\\ &= \int_a^b \left(\int_c^d f(x,y)dy\right) dx.\end{aligned}$$

The significance of Fubini's theorem is that it allows us to evaluate the double integral as an iterated integral, which allows for explicit calculation via the Fundamental Theorem of Calculus for one dimensional function as given in Calculus 1.

3 Double integral over general regions

We do not have to restrict to a rectangle $[a,b]\times[c,d]$ to discuss the double integral of $f(x,y)$ over a more general region. For example consider the circle of radius 1 around $(0,0)$. It can be described as the collection of (x,y) such that

$$-\sqrt{1-y^2} \le x \le \sqrt{1-y^2}, -1 \le y \le 1$$

or

$$-\sqrt{1-x^2} \leq y \leq \sqrt{1-x^2}, -1 \leq x \leq 1.$$

Because in iterated integrals we first integrate in one variable (say x) treating the other variable as fixed (say y), the circle makes sense in the iterated integrals as well. Specifically we can calculate

$$\int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} f(x,y) dx dy$$

or

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} f(x,y) dy dx.$$

Note that we can also define a double sum of $f(x,y)$ over the partition of a circle. To partition the circle, we first perform the partition over the rectangle $[-1,1] \times [-1,1]$ as described in the previous section. We then only select the sub-rectangles that has non-empty intersection with the circle. This also defines (abstractly) the double intergral over the circle $C_{((0,0),1)}$:

$$\iint_{C_{((0,0),1)}} f(x,y) dx dy.$$

Intuitively again we expect that these three integrals have the same value because they are all different approaches to evaluate the volume under of $f(x,y)$ over the region. The Fubini's theorem stated above is indeed still true over more general region. We will state the theorem at the end of the section. Before that, we want to discuss the iterated integral over a more general region than the circle. The reason the circle is still a slightly special region is because it is of the form:

$$h_1(y) \leq x \leq h_2(y), -c \leq y \leq d \tag{4}$$

or

$$g_1(x) \leq y \leq g_2(x), -a \leq x \leq b. \tag{5}$$

We note that not all regions have the form either of (4) or (5). For example consider the region that is the intersection of three curves:

$$\begin{aligned} y_1(x) &= x^2 \\ y_2(x) &= 4 - x^2 \\ (y_3 - 2)^2 + x^2 &= 2. \end{aligned}$$

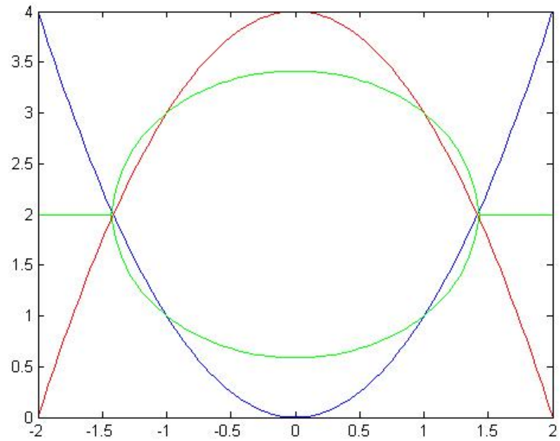

Figure 3.1: Intersection of three curves y_1, y_2, y_3

We also emphasize that a region might be of form (4) but not of (5) or vice versa. For example consider the region that is the intersection of two curves:

$$\begin{aligned} y_1(x) &= x^2 \\ y_2(x) &= 4 - x^2. \end{aligned}$$

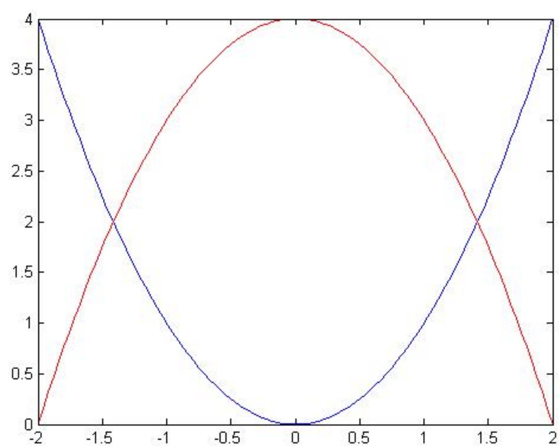

Figure 3.2: Intersection of two curves y_1, y_2

Yet abstractly, we can still define the iterated integrals over all such region. We only need to understand that we integrate first, say in x, over one boundary point of the region to another boundary point of the region, and then repeat the process in y.

Now let $\mathcal{R}$ be a general closed, bounded and convex region. We denote

$$\int \left(\int f(x,y)dx \right) dy \Big|_{\mathcal{R}}$$
$$\int \left(\int f(x,y)dy \right) dx \Big|_{\mathcal{R}}$$

as the two possible iterated integrals over $\mathcal{R}$ and

$$\iint_{\mathcal{R}} f(x,y)dxdy$$

as the double integral over $\mathcal{R}$. Then we have the following result

Theorem 3.1. *(Fubini) Let $\mathcal{R}$ be a closed, bounded and convex region. Let $f(x,y)$ be such that*

$$\iint_{\mathcal{R}} |f(x,y)|dxdy$$

exists then

$$\begin{aligned} \iint_{\mathcal{R}} f(x,y)dxdy &= \int \left(\int f(x,y)dx \right) dy \Big|_{\mathcal{R}} \\ &= \int \left(\int f(x,y)dy \right) dx \Big|_{\mathcal{R}}. \end{aligned}$$

4 Triple integrals

4.1 Triple integral over a rectangular region

Consider $f(x,y,z)$ over the region $\mathcal{R} = [a,b] \times [c,d] \times [p,q]$. That is the region such that $a \leq x \leq b, c \leq y \leq d, p \leq z \leq q$. We define the Riemann sum of f over the region $\mathcal{R}$ similar to the case of the double integral:

$$\sum_{i,j,k} f(x_i, y_j, z_k)(x_{i+1} - x_i)(y_{j+1} - y_j)(z_{k+1} - x_k),$$

where

$$a = x_0 < \cdots < x_L = b; c = y_0 < \cdots < y_M = d; q = z_0 < \cdots < z_N = q$$

is a partition of the region $\mathcal{R}$. If f is regular (e.g. continuous) then this triple sum converges to a limit as the size of the partition goes to 0. We define this limit as the triple integral of f over $\mathcal{R}$:

$$\iiint_{\mathcal{R}} f(x,y,z)\, dx\, dy\, dz.$$

We can also define the itegrated triple integrals:

$$\iint_{[c,d]\times[p,q]} \left(\int_a^b f(x,y,z)dx \right) dydz$$
$$\iint_{[a,b]\times[p,q]} \left(\int_c^d f(x,y,z)dy \right) dxdz$$
$$\iint_{[a,b]\times[c,d]} \left(\int_p^q f(x,y,z)dz \right) dxdy.$$

Note that after we integrate out one variable (say x) then the triple integral integrated in x becomes a double integral. Then we can apply the results we learned in the previous sections to handle the double integral. We also have, via Fubini's Theorem, that the iterated triple integral is equal to the triple integral if f is integrable over the region $\mathcal{R}$:

$$\begin{aligned}\iiint_{\mathcal{R}} f(x,y,z)\, dx\, dy\, dz &= \iint_{[c,d]\times[p,q]} \left(\int_a^b f(x,y,z)dx \right) dydz \\ &= \iint_{[a,b]\times[p,q]} \left(\int_c^d f(x,y,z)dy \right) dxdz \\ &= \iint_{[a,b]\times[c,d]} \left(\int_p^q f(x,y,z)dz \right) dxdy.\end{aligned}$$

4.2 Triple integral over a general region

We can also discuss the triple integral of f over a more general region that has the form:

$$(x,y) \in \mathcal{D}, g_1(x,y) \le z \le g_2(x,y), \tag{6}$$

where $\mathcal{D}$ is a general region in the $xy-$plane that we discuss in section (3). In this case the triple integral is equal to the iterated integral first over z, then over the region $\mathcal{D}$:

$$\iiint_{\mathcal{R}} f(x,y,z)\, dx\, dy\, dz = \iint_{\mathcal{D}} \left(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dz \right) dxdy.$$

Note that the form (6) can also be generalized to

$$(y,z) \in \mathcal{D}, g_1(y,z) \le x \le g_2(y,z),$$

or

$$(x,z) \in \mathcal{D}, g_1(x,z) \leq y \leq g_2(x,z).$$

In the first case we just have to first integrate in x, and then in y, z and in the second case first in y, and then in x, z.