Vectors in three dimensions

Math 251

August 28, 2015

1 Overview of Calculus

1.1 What is Calculus?

1.1.1 The continuous and the infinite

One may say Calculus is the study of the continuous and the infinity. That is the study of the very small, and the very big.

The easiest way to think of the continuous is by a collection of numbers with the property that between any two numbers x, z, we can always find a *distinct* number y. That is given x, z, we can find y such that $x < y < z$.

The most popular example of continuous objects are space and time. Time is a one dimensional continuous object and space (the one we physically live in) is a three dimensional continuous object. We will make precise the meaning of dimension later in the course.

We also think of space and time as *infinite*. That is given any point in time (or in space), we can always add a fixed quantity and get to a point beyond the given one. In one dimension we say given any number x and a fixed distance L, we can always find a distinct number y so that $x + L < y$.

1.1.2 Calculus

It is clearly not very interesting just to have space and time. We do things in space and time. That is we have objects, say our position x as we are driving on a road, as a function of time t. Symbolically we write $x(t)$. We can also think of the height z of where we are on a mountain , as a function of our GPS coordinates on the earth x, y. Symbolically we write $z(x, y)$.

In the first example, we can ask how to describe x in that *instantaneous* moment at $t = t_0$ when we hit the gas pedal. Certainly x will increase as t increase (if we take the direction of our motion as positive). But we want to be more precise: how fast is x increasing? We can measure it by

$$v(\Delta t) := \frac{x(t) - x(t_0)}{t - t_0},$$

where $\Delta t = t - t_0$. Clearly to measure the *instantaneous* effect we need Δt to be small, as small as possible in fact. That is we want to ask what happens to $v(\Delta t)$ as Δt goes to 0. So here we think of v, the average velocity, as a function of Δt and we want to obtain the limiting behavior of v at $\Delta t = 0$. Notice this question makes sense implicitly because time is *continuous*. If v is only defined for certain values of Δt, for example the integers, that is we can only talk about $v(0), v(1), v(2)$ etc. then there is no point in asking about the limiting behavior of v.

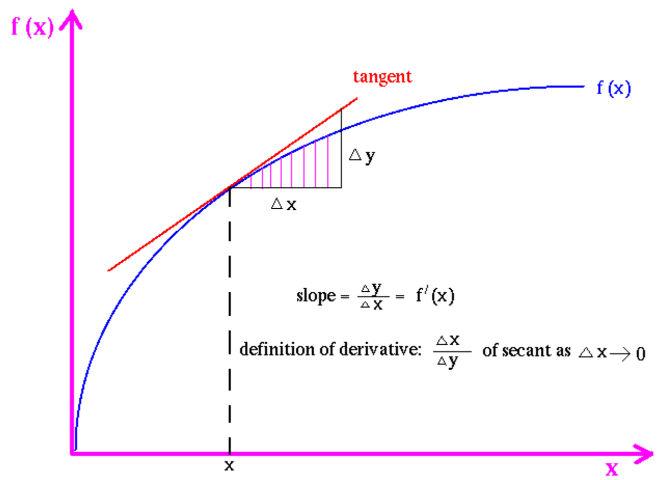

Figure 1.1: Derivative as a limit

Calculus makes precise this notion of limiting behavior (or just limit) by the Epsilon-Delta definition. We can easily think of an example where we need to make precise the notion of limiting behavior at infinity. Let $x(t)$ model the population of bacteria in a test tube as a funtion of time, say $x(t) := \frac{t+1}{t}$. We leave the test tube in the incubation for a long time (long in terms of the life of bacteria). We then ask approximately what is $x(t)$ after this "long" time. In other words, we ask what is $\lim_{t\to\infty} \frac{t+1}{t}$?

Remark: Continuity and infinity do not have to be mutually exclusive, nor do they have to be together. The integers are infinite but not continuous, the interval $(0, 1)$

is continuous but finite, the list of numbers $\{1, 2, \cdots, 10\}$ is finite and discrete (that is non-continuous) and finally the half line $(0, \infty)$ is both continuous and infinite.

As soon as something is infinite *or* continuous then it is potentially within the realm of Calculus study. Thus we study the maxima and minima of a function on a finite (but continuous) interval $(0, T)$. We discuss the limit of a sequence, , that is a function over the integers, $x(n) = \frac{1+n^2}{n}$ as $n \to \infty$. By looking at the partial sum $s(N) := \sum_{n=0}^{N} \frac{1}{n}$ we can look at a series as a sequence and thus discuss the limiting behavior of a series as we discuss sequences: $\lim_{N\to\infty} \sum_{n=0}^{N} \frac{1}{n}$.

1.2 Calculus 3 versus Calculus 1,2

Calculus 1,2 are usually referred to as *single variable* calculus and Calculus 3 is usually referred to as *multi-variable* Calculus. Thus the difference is clear: Calculus 1 and 2 studies functions of one variable while Calculus 3 studies functions of many variables. Several remarks are in order. First while it is true that the calculus techniques for functions of two variables, i.e. $z = f(x, y)$, are almost the same as the techniques for functions of more than two variables i.e. $y = f(x_1, x_2, \cdots, x_n), n > 2$, we will limit ourselves to functions of *two* variables in Calculus 3. One benefit of this restriction is the ease of demonstration (i.e. drawing graphs and illustrations in 3-d).

Some of the common examples of function of *one* variable that arise in physics are functions that depend on time. Thus for a particle we can study its position $s(t)$, its velocity $v(t)$ and its accelaration $a(t)$. On the other hand, if we have a function of two variables, one of the most immediately intuitive examples that come to mind is the height z (of a graph, of a mountain etc.) as a function of the (x, y). coordinates. Thus 3-d geometry is intimately connected to Calculus 3 and Calculus 3 techniques can be used to understand certain properties of 3 dimensional objects. This suggests that we start the studying of Calculus 3 with 3-d geometry, beginning with the coordinate system and vectors.

2 Three-dimensional vectors

2.1 Three dimensional Cartesian coordinate system

You should be very familiar with the Cartesian coordinate system on a plane, where a point is represented by its horizontal x and vertical y coordinates. For example the origin O is represented by $(0, 0)$. If we add one more coordinate z to represent the height, then we have a complete description of a point in three dimensions, the world we live in. For example, the origin O is now represented as $(0, 0, 0)$ where the last coordinate also shows that the height of O is at 0.

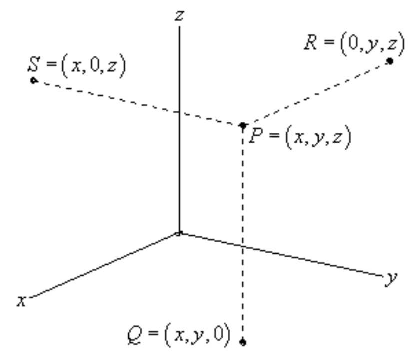

Figure 2.1: Three dimensional Cartesian coordinate system

2.2 Vectors in 3-d

Imagine we have two points in 3-d $P = (1, 2, 3)$ and $Q = (4, 0, 2)$. To move from P to Q we need to move 3 units to the east, 2 units south and 1 unit down. This information can easily be captured by the triple $\langle 3, -2, -1\rangle$. Note that we implicitly take *east, north and up* as positive directions here. Indeed it is immediate to see that all movements from one point to another in 3-d can be captured in such way, and we do not need to explicitly identify P and Q, if we are only interested in the *direction* and *magnitude* of the movement. The mathematical object that capture such information (direction and magnitude) is a vector.

Definition 2.1. *A three-dimensional vector is an ordered list of three real numbers* $\mathbf{x} = \langle x_1, x_2, x_3\rangle$. *The set of all such vectors is denoted by* $\mathbb{R}^3$. x_i *will be referred to as the i-th component of the vector* $\mathbf{x}$.

Notation: We will use a bold-faced, lower case letter to denote a generic vector in this course. For example we say let $\mathbf{v}$ be a vector in 3-d. What we mean is there is a set of real numbers $\langle v_1, v_2, v_3\rangle$ that completely describes $\mathbf{v}$, which we do not specify at the moment. We will use regular, lower case letter, s, t, x, y to denote a scalar, that is, a real number.

Remark 1: By an ordered list we mean that the vector $\langle 1, 2, 3\rangle$ is different from the vector $\langle 2, 1, 3\rangle$, for example. It is obvious why this is the case.

Remark 2: If we consider the set of points whose third coordinate is always a constant, for example $(x, y, 1)$ then clearly these points belong to a plane. Similarly, if we consider the set of vectors whose third component is always a constant, for example $\langle x, y, 0\rangle$ then we look at the set of movements starting from a point on a horizontal plane and never leaving that plane (since the vertical movement is 0 in magnitude). Thus the two-dimensional world is naturally embedded in the 3-d world and all of the concepts we develop in 3-d can easily be reduced down to 2-d.

Remark 3 - Vector versus point: You may note that the representation of a vector is very similar to the representation of a point. To distinguish them, we use parentheses () for point and $\langle\rangle$ for vector. Mathematically, they are also very similar in the following sense. Given two points PQ we can come up with a vector that points from P to Q. On the other hand, given a vector $\mathbf{v}$, we can simply think of the starting point P as the origin $(0, 0, 0)$. Thus the set of vectors $\mathbf{v}$ in $\mathbb{R}^3$ also corresponds to the set of points in $\mathbb{R}^3$ and vice versa. However, conceptually we mean different things by these two: a point does not have magnitude and direction, while a vector does.

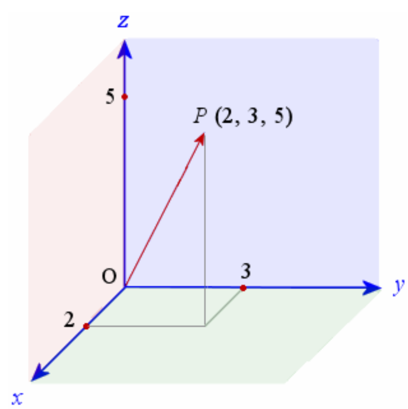

Figure 2.2: A vector from the origin to P

Remark 4 - Parallel vectors: Suppose that we start at the origin (0,0,0) and follow a certain vector to arrive at (1,2,3). Also suppose that we start at (1,1,1) and follow a certain vector to arrive at (2,3,4). These are movements happening at different locations, but note that in both cases, we move 1 unit east, 2 units north and 3 units up. Thus as far as *direction and magnitude* of movements are concerned, these are the same type of movement. On the other hand, if you draw these two movement vectors you will see that they are parallel. Thus it is convenient to identify all paralel vectors as one single vector.*By default*, we think of this vector as starting at the origin (0,0,0).

2.3 Vector magnitude

When we move from (0,0,0) to (1,1,1), by Pythagorean theorem, it is easy to check that we have moved a distance of $\sqrt{3}$ unit. Equivalently, we say the magnitude of the vector $\langle 1, 1, 1\rangle$ is $\sqrt{3}$. Indeed we have the following easy Lemma

Lemma 2.2. *The magnitude* $\|\mathbf{v}\|$ *of a vector* $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ *is*

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}.$$

Proof. Left to the reader.

3 Vector algebra

3.1 Scalar multiplication

Imagine we start at the origin (0,0,0) and follow the vector $\langle 1, 2, 3\rangle$ to arrive at (1,2,3). Once there we decide to follow $\langle 1, 2, 3\rangle$ again. Then we arrive at (2,4,6). But clearly the whole trip can be viewed as starting at (0,0,0) and follow $2 \times \langle 1, 2, 3\rangle$. Alternatively, we can start at (0,0,0) and follow $\langle 4, 6, 6\rangle$. All of these descriptions land us on the same final point.

Of course we do not have to follow an integer multiple of a vector. We can follow 3/2, and even by a stretch of imagination $\sqrt{2}$ times the vector $\langle 1, 2, 3\rangle$. All that we mean here is we follow the same *direction* as $\langle 1, 2, 3\rangle$ but in the magnitude t times $\|\langle 1, 2, 3\rangle\|$ where $t = 3/2, \sqrt{2}, \cdots$. The last thing to mention is if t is negative, we follow the *opposite* direction as that of $\langle 1, 2, 3\rangle$. Thus we have

Definition 3.1. *Let* t *be a real number and* $\mathbf{v} = \langle v_1, v_2, v_3\rangle$ *a vector in* $\mathbb{R}^3$. *Then* $t\mathbf{v}$ *is also a* $\mathbb{R}^3$ *vector such that*

$$t\mathbf{v} = \langle tv_1, tv_2, tv_3\rangle$$

The following is an immediate result that is left to the reader to prove: for all $t \in \mathbb{R}$ and $\mathbf{v} \in \mathbb{R}^3$

$$\|t\mathbf{v}\| = |t|\|\mathbf{v}\|.$$

3.2 Vector addition

Imagine we start at the origin (0,0,0) and follow the vector $\langle 1, 2, 3\rangle$ to arrive at (1,2,3). Once there we decide to follow $\langle 2, -1, -2\rangle$. Then we arrive at (3,1,1). Clearly the whole trip can also be viewed as starting at (0,0,0) and follow the vector $\langle 3, 1, 1\rangle$, which can be view as the addition of $\langle 1, 2, 3\rangle$ and $\langle 2, -1, -2\rangle$. Thus we have

Definition 3.2. *Let* $\mathbf{x} = \langle x_1, x_2, x_3\rangle$ *and* $\mathbf{y} = \langle y_1, y_2, y_3\rangle$ *be two vectors in* $\mathbb{R}^3$*. Then* $\mathbf{x} + \mathbf{y}$ *is also a vector in* $\mathbb{R}^3$ *such that*

$$\mathbf{x} + \mathbf{y} = \langle x_1 + y_1, x_2 + y_2, x_3 + y_3\rangle.$$

The following Lemma is also immediate and left to the reader to prove

Lemma 3.3. *We have for all* $t \in \mathbb{R}, \mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^3$

$$\begin{aligned} \mathbf{u} + \mathbf{v} &= \mathbf{v} + \mathbf{u} \\ \mathbf{u} + (\mathbf{v} + \mathbf{w}) &= (\mathbf{u} + \mathbf{v}) + \mathbf{w} \\ t(\mathbf{u} + \mathbf{v}) &= t\mathbf{u} + t\mathbf{v}. \end{aligned}$$

3.2.1 Parallelogram rule

Our definition states that if we add $\langle 1, 2, 3\rangle$ and $\langle 2, -1, -2\rangle$ we get $\langle 3, 1, 1\rangle$. Drawing all these three vectors as starting at the origin (0,0,0), how do we get $\langle 3, 1, 1\rangle$ from $\langle 1, 2, 3\rangle$ and $\langle 2, -1, -2\rangle$? The answer is the parallelogram rule, as in the following picture

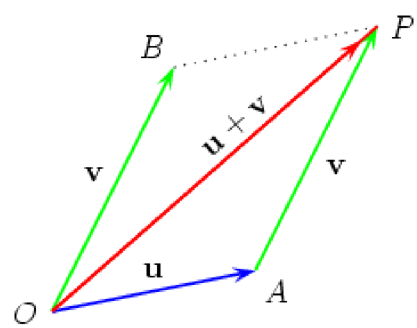

Figure 3.1: Parallelogram rule

The interpretation of adding vectors using the parallelogram rule is most suitable for physical forces, where we think of **u** and **v** as two forces acting on the same object, some point mass, at the origin. Clearly we cannot interpret the net effect of these forces as sequential movements. The resulting force is $\mathbf{u} + \mathbf{v}$.

3.2.2 Unit vector and standard basis vectors

We have the following basic observation: the magnitude of the vector $\frac{\mathbf{v}}{\|\mathbf{v}\|}$ is 1. That is

$$\|\frac{\mathbf{v}}{\|\mathbf{v}\|}\| = 1.$$

Any vector whose magnitude is 1 is called a *unit* vector. The above observation shows us how to obtain, for any vector $\mathbf{v}$, a unit vector pointing in the *same direction* as $\mathbf{v}$. Unit vector is useful when all we care about is *direction* but not the magnitude of a vector.

In particular, there are three important unit vectors that arise in our study of 3-d geometry:

$$\begin{aligned} \mathbf{i} &= \langle 1, 0, 0 \rangle \\ \mathbf{j} &= \langle 0, 1, 0 \rangle \\ \mathbf{k} &= \langle 0, 0, 1 \rangle. \end{aligned}$$

They correspond to the east, north and up directions. We refer to these three vectors as *the standard basis vectors* in $\mathbb{R}^3$. They are standard because of the elementary property that *any* vector $\mathbf{v}$ can be written as a summation of scalar multiplication of the standard basis. Indeed:

$$\mathbf{v} = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}.$$

Terminology: We will refer to "summation of scalar multiplication of a colletion of vectors" as a *linear combination* of those vectors.

3.3 Triangle inequality

From Figure 3.1 we can easily see that $\mathbf{u}, \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$ form a triangle. We thus have the following triangle inequality, which is another way to state that the straightline is the shortest path in between two points:

$$\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

We will prove this inequality once we develop the notion of dot product for vectors, which is closely related to the cosine rule for triangle.

3.4 Parametric representation of a line

The first application we have for vectors is to represent a line. Traditionally, we think of a line as defined by two points P_0 and P and extend indefinitely on either side of P_0 or P. We say two points uniquely determine a line.

Alternatively, given a point $P_0 = (x_0, y_0, v_0)$ and a vector $\mathbf{v} = \langle v_1, v_2, v_3 \rangle$ (pointing to P perhaps) we can start at P and follow either $\mathbf{v}$ or $-\mathbf{v}$ to obtain the same line as above. In other words, a point and a vector also uniquely determine a line. Let's denote this line as $\mathcal{L}$.

Any point P on $\mathcal{L}$ can be represented by a vector as

$$\overrightarrow{OP} = \langle x_0, y_0, v_0 \rangle + t\langle v_1, v_2, v_3 \rangle,$$

where O is the origin and t is a to be dertemined parameter, depending on where P is. Alternatively, we can just write

$$P = (x_0 + tv_1, y_0 + tv_2, z_0 + tv_3).$$

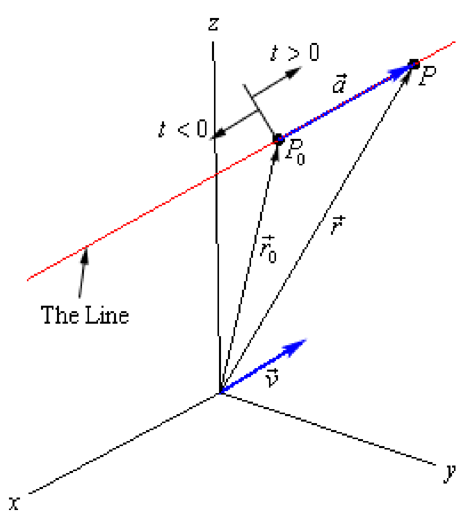

Figure 3.2: Parametric representation of a line

Finally, given two points P, Q, to describe the line that goes through P and Q we simply let $\mathbf{v} = \overrightarrow{PQ}$ and follow the above formula.