Math 485
Fall 2014
Final exam
12/16/2014

Name (Print): ____________________

This exam contains 8 pages (including this cover page) and 8 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note (1 sided) and a scientific calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. For example, in question involved the multi-period binomial model, I would like to see how you derive the no arbitrage price, say by displaying the tree with all the nodes filled out if the situation is appropriate.
- If the answer involves the probability of a well-known distribution, says the Normal(0,1), **you can leave the answer in the form** $P(Z > x)$ **or** $P(Z < x)$ where x is a number you found from the problem.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	30	
2	30	
3	20	
4	20	
5	20	
6	20	
7	20	
8	40	
Total:	200	

1. (30 points) Let $V_T = (S_T - K)^+$. Derive the Black-Scholes formula for V_0.

   Ans: See class notes.

2. (30 points) Derive the Black-Scholes PDE for $v(t, S_t) = V_t$ where $V_T = f(S_T)$.

   Ans: See class notes.

3. Consider the multiperiod model where $S_0 = 81$, $u = 4/3, d = 2/3, \Delta_T = 1$ and $r = 0$.

   (a) (10 points) Consider the lookback option with expiration time $N = 4$ on S. That is $V_4 = \max_{i=0,\cdots,4} S_i$. Find V_1.

   Ans:
   $q = \frac{1-d}{u-d} = 1/2$.

$$\begin{aligned} V_1(u) &= [256 + (192 + 144 + 128) \\ &\quad +(144 + 108 + 108) + 108]\frac{1}{2^3} \\ &= 148.5 \end{aligned}$$

$$\begin{aligned} V_1(d) &= [128 + (96 + 72 + 64) \\ &\quad +(72 + 54 + 54) + 54] \\ \frac{1}{2^3} &= 74.25 \end{aligned}$$

   (b) (10 points) Consider the up and in call option on S with strike $K = 120$, barrier $L = 150$ and expiration $N = 4$. Find V_1.

$$\begin{aligned} V_1(u) &= [(256 - 120) + (128 - 120)]\frac{1}{2^3} = 18 \\ V_1(d) &= 0. \end{aligned}$$

4. Let S_t follow the Black-Scholes model:

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Consider a Euro-style derivative with expiration T that pays $\log(S_T)$ if $\log(S_T) > K$ and 0 otherwise. That is

$$\begin{aligned} V_T &= \log(S_T) \text{ if } \log(S_T) > K \\ &= 0 \text{ otherwise .} \end{aligned}$$

(a) (10 points) Find V_0 for the above derivative.
Ans: $\log(S_T) > K$ if and only if $W_T > \frac{K - \log(S_0) - (r - 1/2\sigma^2)T}{\sigma}$.
Denote $d = \frac{K - \log(S_0) - (r - 1/2\sigma^2)T}{\sigma}$, we have

$$V_0 = E(e^{-rT} V_T) = e^{-rT}\Big[(\log(S_0) + r - 1/2\sigma^2)TP(W_T > d) + \sigma E(W_T \mathbf{1}_{W_T > d})\Big].$$

$$\begin{aligned} E(W_T \mathbf{1}_{W_T > d}) &= \frac{1}{\sqrt{2\pi T}} \int_d^\infty x e^{-\frac{x^2}{2T}} dx \\ &= \frac{T}{\sqrt{2\pi T}} e^{-\frac{d^2}{2T}}. \end{aligned}$$

Therefore,

$$V_0 = e^{-rT}\Big[(\log(S_0) + r - 1/2\sigma^2)T(1 - N(\frac{d}{\sqrt{T}})) + \sigma \frac{T}{\sqrt{2\pi T}} e^{-\frac{d^2}{2T}}\Big]$$

(b) (10 points) Find $V_t, 0 \le t \le T$ for the above derivative.

$$V_t = e^{-r(T-t)}\Big[(\log(S_t) + r - 1/2\sigma^2)(T-t)(1 - N(\frac{d(t)}{\sqrt{T-t}})) + \sigma \frac{T-t}{\sqrt{2\pi(T-t)}} e^{-\frac{d^2}{2(T-t)}}\Big],$$

where

$$d(t) = \frac{K - \log(S_t) - (r - 1/2\sigma^2)(T-t)}{\sigma}$$

5. Let S_t follow the Black-Scholes model:

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

Consider a Euro-style derivative with expiration T that pays K if $S_T < K$ and 0 otherwise. That is

$$\begin{aligned} V_T &= K \text{ if } S_T < K \\ &= 0 \text{ otherwise .} \end{aligned}$$

(a) (10 points) Find V_t for the above derivative.
Ans:

$$V_t = Ke^{-r(T-t)}(1 - N(d_2(t))),$$

where

$$d_2(t) = \frac{(r - 1/2\sigma^2)(T-t) - \log\left(\frac{K}{S_t}\right)}{\sigma\sqrt{T-t}}.$$

(b) (10 points) Let $v(t,x)$ be such that $v(t,S_t) = V_t$ for the V_t you find in part a. Verify that $v(t,x)$ satisfies the Black-Scholes PDE.
Ans:

$$v(t,x) = Ke^{-r(T-t)}(1 - N(d_2(t,x))),$$

where

$$d_2(t,x) = \frac{(r - 1/2\sigma^2)(T-t) - \log\left(\frac{K}{x}\right)}{\sigma\sqrt{T-t}}.$$

Therefore

$$\begin{aligned} -rv &= -Kre^{-r(T-t)}(1 - N(d_2(t,x))) \\ v_t(t,x) &= Kre^{-r(T-t)}(1 - N(d_2(t,x))) - e^{-r(T-t)}\Phi_Z(d_2(t,x))\frac{(r - 1/2\sigma^2)}{2\sigma\sqrt{T-t}} \\ v_x(t,x) &= -Ke^{-r(T-t)}\Phi_Z(d_2(t,x))\frac{1}{x\sigma\sqrt{T-t}} \\ v_{xx}(t,x) &= Ke^{-r(T-t)}\Phi_Z(d_2(t,x))\frac{1}{x^2\sigma\sqrt{T-t}} \end{aligned}$$

We see that

$$-rv + v_t + rxv_x + 1/2\sigma^2 x^2 v_{xx} = 0.$$

6. Let $r = 0.03, \sigma = 0.2, T = 1, S_0 = 2000$ and $K = 2000$. Consider a European Call option on S with strike K and expiration T.

   (a) (10 points) Compute the Black-Scholes price for V_0.

$$\begin{aligned} d_1 &= \frac{(0.03 + 1/2(0.2)^2)1 - \log(2000/2000)}{0.2\sqrt{1}} = 0.25 \\ d_2 &= \frac{(0.03 - 1/2(0.2)^2)1 - \log(2000/2000)}{0.2\sqrt{1}} = 0.05. \end{aligned}$$

   Thus

$$V_0 = 2000N(0.25) - 2000e^{-.03}N(0.05) = 188.3$$

   (b) (10 points) Compute the Binomial approximation to the Black-Scholes price with the number of steps $n = 5$.

   Ans: We use the Binomial model with $X_k = 1 + r(t_{k+1} - t_k) + \sigma\sqrt{t_{k+1} - t_k}Y_k$, where $Y_k = \pm 1$ with probability 1/2. Plug in , we have

$$\begin{aligned} X_k &= 1.0954 \text{ with probability } \frac{1}{2} \\ &= .9045 \text{ with probability } \frac{1}{2}. \end{aligned}$$

   Then

$$V_0 = e^{-.03}(1154.23 + 5 \times 604.53 + 10 \times 150.62)\frac{1}{2^5} = 172.34.$$

7. Consider a market with 2 risky assets S^1, S^2 and the interest rate is 0. We have $S_0^1 = 2, S_0^2 = 3$. There are 3 outcomes for S^1, S^2 at time T, denoted by $\omega_1, \omega_2, \omega_3$. Suppose that $S_T^1(\omega_1) = 3, S_T^1(\omega_2) = 2, S_T^1(\omega_3) = 1$ and $S_T^2(\omega_1) = 4, S_T^2(\omega_2) = S_T^2(\omega_3) = 1$.

   (a) (10 points) Is the market abitrage free? If yes, explain why. If no, produce an arbitrage portfolio.

   Ans: The market is not arbitrage free. Let π be a portfolio with 3 shares of S^1 and shorting 2 shares of S^2. Then $\pi_0 = 0$. But

$$\begin{aligned}
\pi_T(\omega_1) &= 9 - 8 = 1 \\
\pi_T(\omega_2) &= 6 - 2 = 4 \\
\pi_T(\omega_3) &= 3 - 2 = 1
\end{aligned}$$

   So π is an arbitrage opportunity.

   (b) (10 points) Is the market complete? Explain.

   The market is complete, because the equation

$$\begin{bmatrix} 3 & 4 & 1 \\ 2 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} \Delta_1 \\ \Delta_2 \\ y \end{bmatrix} = \begin{bmatrix} V_T(\omega_1) \\ V_T(\omega_2) \\ V_T(\omega_3) \end{bmatrix}$$

   always has a solution by observing that the LHS matrix can be reduced to

$$\begin{bmatrix} 3 & 4 & 1 \\ 1 & 3 & 0 \\ 2 & 3 & 0 \end{bmatrix}$$

   and from there we see that it is invertible.

8. Let S_t follow the Black-Scholes model:

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dW_t \\ S_0 &= x. \end{aligned}$$

(a) (10 points) Let $V_0^C(K)$ denote the price of a call option on S with expiration T and strike K. Use Black-Scholes formula to compute $\lim_{K\to\infty} V_0^C(K)$.

Ans:

$$\begin{aligned} d_1 &= \frac{(r+1/2\sigma^2)T - \log(K/S_0)}{\sigma\sqrt{T}} \\ d_2 &= \frac{(r-1/2\sigma^2)T - \log(K/S_0)}{\sigma\sqrt{T}} \end{aligned}$$

As $K \to \infty, d_1, d_2 \to -\infty$. We still need to decide

$$\lim_{K\to\infty} KN(d_2) = \lim_{K\to\infty} \frac{N(d_2)}{1/K}.$$

Using L' Hospital's rule we get

$$\lim_{K\to\infty} \frac{N(d_2)}{1/K} = -\frac{\phi_Z(d_2)\frac{S_0}{K^3}}{1/K^2} = 0,$$

as ϕ_Z is a bounded function. Thus

$$V_0^C = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

goes to 0 as $K \to \infty$.

(b) (10 points) Let $V_0^P(K)$ denote the price of a put option on S with expiration T and strike K. Compute $\lim_{K\to\infty} V_0^P(K)$ (Hint: You can use either the put call parity or the Black-Schole formula here).

By the Put-Call parity:

$$V_0^P = V_0^C - (S_0 - Ke^{-rT}).$$

Therefore, as V_0^C goes to 0, V_0^P goes to ∞ as $K \to \infty$.

(c) (10 points) Let $V_0^C(K)$ denote the price of a call option on S with expiration T and strike K. Use Black-Scholes formula to compute $\lim_{T\to\infty} V_0^C(T)$.

As $T \to \infty, d_1 \to \infty$. Therefore,

$$V_0^C = S_0 N(d_1) - Ke^{-rT} N(d_2)$$

goes to S_0 as $T \to \infty$.

(d) (10 points) Let $V_0^P(T)$ denote the price of a put option on S with expiration T and strike K. Compute $\lim_{T\to\infty} V_0^P(T)$ (Hint: You can use either the put call parity or the Black-Schole formula here).

Again, by the Put-Call parity:

$$V_0^P = V_0^C - (S_0 - Ke^{-rT}).$$

As V_0^C goes to S_0, V_0^P goes to 0 as T goes to ∞.