Math 485
Fall 2014
Midterm 2
11/20/2014

Name (Print): ______________________________

This exam contains 6 pages (including this cover page) and 5 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note (1 sided) and a scientific calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. For example, in question involved the multi-period binomial model, I would like to see how you derive the no arbitrage price, say by displaying the tree with all the nodes filled out if the situation is appropriate.
- If the answer involves the probability of a well-known distribution, says the Normal(0,1), **you can leave the answer in the form** $P(Z > x)$ **or** $P(Z < x)$ where x is a number you found from the problem.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	20	
4	20	
5	20	
Total:	100	

1. Consider the multiperiod model where $S_0 = 81$, $u = 4/3, d = 2/3, \Delta_T = 1$ and $r = 0.1$ per period (so that the discount factor is $e^{-r\Delta_T} = e^{-r}$ per period).

   (a) (10 points) Compute the price of V_0 the American put option with strike price $K = 100$ and expiration time $n = 3$.
   The risk neutrl probability is $\frac{e^r - d}{u-d} \approx .66$.
   The value of the option is approximately 19.73.

   (b) (10 points) Find the optimal (random) exercise time τ^* and check that

   $$V_0 = E^Q\Big(e^{-r\tau^*}(K - S_{\tau^*})^+\Big).$$

   The distribution of the stopping time is as followed:

   $$\begin{aligned}
   \tau^*(uuu) &= 3\\
   \tau^*(uud) &= 3\\
   \tau^*(udu) &= 2\\
   \tau^*(udd) &= 2\\
   \tau^*(duu) &= 1\\
   \tau^*(dud) &= 1\\
   \tau^*(ddu) &= 1\\
   \tau^*(ddd) &= 1.
   \end{aligned}$$

   Correspondingly,

   $$\begin{aligned}
   (K - S_\tau^*)^+(uuu) &= 0\\
   (K - S_\tau^*)^+(uud) &= 4\\
   (K - S_\tau^*)^+(udu) &= 28\\
   (K - S_\tau^*)^+(udd) &= 28\\
   (K - S_\tau^*)^+(duu) &= 46\\
   (K - S_\tau^*)^+(dud) &= 46\\
   (K - S_\tau^*)^+(ddu) &= 46\\
   (K - S_\tau^*)^+(ddd) &= 46.
   \end{aligned}$$

   With discount

   $$\begin{aligned}
   e^{-r\tau^*}(K - S_\tau^*)^+(uuu) &= 0\\
   e^{-r\tau^*}(K - S_\tau^*)^+(uud) &= 2.96\\
   e^{-r\tau^*}(K - S_\tau^*)^+(udu) &= 22.92\\
   e^{-r\tau^*}(K - S_\tau^*)^+(udd) &= 22.92\\
   e^{-r\tau^*}(K - S_\tau^*)^+(duu) &= 41.62\\
   e^{-r\tau^*}(K - S_\tau^*)^+(dud) &= 41.62\\
   e^{-r\tau^*}(K - S_\tau^*)^+(ddu) &= 41.62\\
   e^{-r\tau^*}(K - S_\tau^*)^+(ddd) &= 41.62.
   \end{aligned}$$

With probability

0	.29
2.96	.15
22.92	.22
41.62	.34

Thus we can compute

$$(2.96)(.15) + (22.92)(.22) + (41.62)(.34) = 19.63,$$

which is close to what we had in part a with some rounding errors.

2. (20 points) Consider the same model as Question 1, that is $S_0 = 81$, $u = 4/3, d = 2/3$ but with interest rate $r = 0$.

   Compute the price of V_0 of a Down and Out European Call Option with barrier $L = 60$, strike price $K = 100$ and expiration time $n = 3$.

   Ans: The risk neutral measure is

$$q = \frac{1-d}{u-d} = \frac{1}{2}.$$

   You can check that the only paths that "survive" the barrier are: uuu, uud, udu. The pay off at each of these are:

$$\begin{aligned} V_3(uuu) &= 192 - 100 = 82 \\ V_3(uud) &= (96 - 100)^+ = 0 \\ V_3(udu) &= (96 - 100)^+ = 0. \end{aligned}$$

   Thus the option price is $\frac{82}{8}$.

3. Let W_t be a Brownian motion. Recall that for all $r < s < t$ $W_t - W_s$ is independent of W_r. Compute the following.

   (a) (10 points) For $s < t, E(W_s W_t)$.
   Ans:

$$E(W_s W_t) = E(W_s(W_s + W_t - W_s)) = E(W_s^2) + E(W_s)E(W_t - W_s) = s.$$

   (b) (5 points) $E(e^{\int_0^T t dW_t - T^2})$.
   Ans: $\int_0^T t dW_t$ has distribution $N(0, \frac{T^3}{3})$. Therefore

$$E(e^{\int_0^T t dW_t}) = e^{\frac{T^3}{6}}.$$

   That is

$$E(e^{\int_0^T t dW_t - T^2}) = e^{\frac{T^3}{6} - T^2}.$$

   (c) (5 points) $E([\int_0^T e^{W_t} dW_t]^2)$.
   Ans:

$$E([\int_0^T e^{W_t} dW_t]^2) = \int_0^T E(e^{2W_t})dt = \int_0^T e^{2t} dt = \frac{e^{2T} - 1}{2}.$$

4. Let W_t be a Brownian motion. Compute

   (a) (10 points) $d(t^2 e^{W_t})$.
   Ans:

$$d(t^2 e^{W_t}). = 2te^{W_t} dt + t^2 e^{W_t} dW_t + \frac{1}{2} t^2 e^{W_t} dt.$$

   (b) (10 points) $d(W_t^2 e^t)$.

Ans:

$$d(W_t^2 e^t) = e^t W_t^2 dt + e^t 2 W_t dW_t + e^t dt.$$

5. After his music debacle, Mr. Solve-alot decided to focus on finance only and got a position with Goldman Sach. He plans to rock the finance world by proposing a model to replace the famous Black-Scholes model. In particular, he proposes the following model for the stock:

$$\begin{aligned} dS_t &= rS_t dt + \sigma dW_t, \\ S_0 &= x. \end{aligned}$$

under the risk neutral measure Q where x, r, σ are constants, r is the interest rate.

(a) (5 points) Compute $d(e^{-rt}S_t)$ where S_t satisfies the above model.
Ans:

$$d(e^{-rt}S_t) = -re^{-rt}S_t dt + e^{-rt}dS_t = e^{-rt}\sigma dW_t.$$

(b) (5 points) Note that the RHS of your answer in part a is free of S_t. Use this to write down an explicit formula for S_t (Your answer may involve an Ito integral and other expression involving x, r but should not have S anywhere on the RHS).
Ans:

$$e^{-rt}S_t - x = \int_0^t e^{-rs}\sigma dW_s.$$

That is

$$S_t = xe^{rt} + \int_0^t e^{r(t-s)}\sigma dW_s.$$

(c) (5 points) Find the distribution of S_t.
Ans: $\int_0^t e^{r(t-s)}\sigma dW_s$ has distribution $N(0, \int_0^t e^{2r(t-s)}\sigma^2 ds) = N(0, \sigma^2 e^{2rt}\frac{1-e^{-2rt}}{2r})$.
Thus S_t has $N(xe^{rt}, \sigma^2\frac{e^{2rt}-1}{2r})$ distribution.

(d) (5 points) Upon proposing his model with his manager, Mr. Solve-alot immediately loses his job. Can you explain why?
Ans: S_t has a positive chance of being a negative number (it has Normal distribution). Thus it cannot be a reasonable model for a stock price.