Math 485
Fall 2014
Midterm 1
10/09/2014

Name (Print): ____________________________

This exam contains 7 pages (including this cover page) and 6 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If the answer involves the probability of a well-known distribution, says the Normal(0,1), **you can leave the answer in the form** $P(Z > x)$ **or** $P(Z < x)$ where x is a number you found from the problem.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	20	
6	20	
Total:	100	

1. (a) (5 points) The time a Math 485 student takes to finish the first midterm is an Exponential(1) random variable (The expectation is 1 hour in this case). There are 25 students in the class, and suppose the time it takes for each one to finish are independent. What is the approximate probability that the average time the students take will exceed 1 hour 15 minutes?

   Ans: Let $\bar{X}$ denote the average time. Then $E(\bar{X}) = 1$ and $Var(\bar{X}) = \frac{1}{25}$. Therefore by CLT

$$P(\bar{X} \geq 5/4) \approx P(Z \geq \frac{5/4-1}{1/5}) = P(Z \geq 5/4).$$

   (b) (5 points) Now suppose the time a Math 485 student takes to finish the first midterm is an Exponential(Y) random variable, where Y is Uniform[1/2,1]. Again there are 25 students in the class, and suppose the time it takes for each one to finish are independent. What is the approximate average time the students take to finish the exam?

   Ans: Let X have distribution Exp(Y), then

$$E(X) = E(E(X|Y)) = E(Y) = 3/4.$$

   By the LLN, the average time they take is 3/4 of an hour.

   (c) (5 points) The time it takes for John Solve-alot to finish his first midterm is a Uniform[0,Y], where Y is a Uniform[3/4,5/4] random variable. What is the probability that Mr. Solve-alot will take less than 1 hour to finish his exam?

Ans: Let X have distribution Uniform(Y). Then

$$f_{X,Y}(x,y) = f_{X|Y}(x|y)f_Y(y) = \frac{2}{y}, 0 \leq x \leq y, 3/4 \leq y < 5/4.$$

It is better to integrate in the order of $dxdy$ here, to get rid of the $1/y$ term.

Therefore (draw a plot of the region if it's not clear to you)

$$\begin{aligned} P(X \leq 1) &= \int_{3/4}^{1}\int_0^y \frac{2}{y}dxdy + \int_1^{5/4}\int_0^1 \frac{2}{y}dxdy \\ &= \int_{3/4}^{1} 2dy + \int_1^{5/4} \frac{2}{y}dy \\ &= 1/2 + 2\log(5/4). \blacksquare \end{aligned}$$

2. (a) (7 points) Consider a forward contract based on 2 underlying assets S^1, S^2 with expiration time T and strike price K, that is at time T, the contract holder can purchase 1 share of S^1 and 1 share of S^2 for the price of K dollars. Assume the market is arbitrage free. What is the price of this forward contract at time 0 if the interest rate is r ?

   Ans: Let V be the value of the forward contract. Then $V_T = S_T^1 + S_T^2 - K$. Let x_i be the number of S^i we hold at time 0 and y be the cash amount. Then we require

$$x_1 S_T^1 + x_2 S_T^2 + ye^{rT} = S_T^1 + S_T^2 - K$$

Which means

$$(x_1 - 1)S_T^1 + (x_2 - 1)S_T^2 = -ye^{rT} - K.$$

The LHS is generally a RV and the RHS is deterministic. Thus the only way the equality holds is for $x_1 = x_2 = 1$ and $y = -Ke^{-rT}$.

Thus

$$V_0 = S_T^1 + S_T^2 - Ke^{-rT}.$$

(b) (8 points) Now suppose additionally that the asset S^1 pays a one time d dollars dividend at time t and asset S^2 pays dividend at rate r_d that is continuously reinvested into S^2. What now is the price of the forward contract described in part a at time 0?

Ans: Set up exactly as part a, except now we require

$$x_1 S_T^1 + x_1 d e^{r(T-t)} + x_2 e^{r_d T} S_T^2 + ye^{rT} = S_T^1 + S_T^2 - K.$$

Which means

$$(x_1 - 1)S_T^1 + (x_2 e^{r_d T} - 1)S_T^2 + ye^{rT} = -x_1 d e^{r(T-t)} - ye^{rT} - K.$$

Again because the LHS is a RV and the RHS is deterministic, we have $x_1 = 1, x_2 = e^{-r_d T}, y = -(K + de^{r(T-t)})e^{-rT}$.

Thus

$$V_0 = S_T^1 + e^{-r_d T} S_T^2 - (K + de^{r(T-t)})e^{-rT}. \blacksquare$$

3. Consider a market with 2 risky assets S^1, S^2 and the interest rate is 0. We have $S_0^1 = 2, S_0^2 = 3$. There are only 2 outcomes for S^1, S^2 at time T, denoted by ω_1, ω_2. Suppose that $S_T^1(\omega_1) = 3, S_T^1(\omega_2) = 1$ and $S_T^2(\omega_1) = 4, S_T^2(\omega_2) = 2$.

(a) (7 points) Is the market abitrage free? If yes, explain why. If no, produce an arbitrage portfolio.

Ans: The market is arbitrage free. The risk neutral measure is $P^Q(\omega_1) = P^Q(\omega_2) = 1/2$.

(b) (8 points) Is the market complete? Explain. Ans: The market is complete as the risk neutral measure is unique.

Alternative answer: consider the system

$$\begin{bmatrix} 3 & 4 & 1 & V_u \\ 1 & 2 & 1 & V_d \end{bmatrix}$$

This is an under-determined system, where the 1st and 2nd columns of the matrix are linearly independent. Thus it is always sovable. Thus every derivative can be replicated and the market is complete. $\blacksquare$

4. Consider the same market in the previous question. That is we have $S_0^1 = 2, S_0^2 = 3$. But now suppose that $S_T^1(\omega_1) = 1, S_T^1(\omega_2) = 3$ and $S_T^2(\omega_1) = 4, S_T^2(\omega_2) = 2$.

(a) (7 points) Is the market abitrage free? If yes, explain why. If no, produce an arbitrage portfolio. Ans: The market is arbitrage free. The risk neutral measure is $P^Q(\omega_1) = P^Q(\omega_2) = 1/2$.

(b) (8 points) Is the market complete? Explain. Ans: The market is complete as the risk neutral measure is unique.

Alternative answer: consider the system

$$\begin{bmatrix} 1 & 4 & 1 & V_u \\ 3 & 2 & 1 & V_d \end{bmatrix}$$

This is an under-determined system, where the 1st and 2nd columns of the matrix are linearly independent. Thus it is always sovable. Thus every derivative can be replicated and the market is complete. ■

5. Consider the multiperiod model with $N = 2, S_0 = 8$. From any given period, the stock can either double its price or half its price in the next period (ask for a picture if this description is not clear). Also suppose the interest rate is 0.

   (a) (10 points) Use the replicating portfolio approach to find the price V_0 of a European call option with strike 10 and expiration $N = 2$ on this stock.

   Ans:

$$\begin{aligned} V_2(uu) &= 22 \\ V_2(ud) &= V_2(dd) = 0. \end{aligned}$$

   We have

$$\pi_2 = \Delta_1 S_2 + (\pi_1 - \Delta_1 S_1).$$

   Plug in uu and ud in the above equation gives

$$\begin{aligned} 22 &= \Delta_1(u)32 + \pi_1(u) - \Delta_1(u)16 \\ 0 &= \Delta_1(u)8 + \pi_1(u) - \Delta_1(u)16. \end{aligned}$$

   Solving gives $\Delta_1(u) = 11/12$ and $V_1(u) = \pi_1(u) = 22/3$.
   Plug in ud and dd in the above equation gives

$$\begin{aligned} 0 &= \Delta_1(d)8 + \pi_1(d) - \Delta_1(d)4 \\ 0 &= \Delta_1(d)2 + \pi_1(d) - \Delta_1(d)4. \end{aligned}$$

   Solving gives $\Delta_1(d) = 0$ and $\pi_1(d) = V_1(d) = 0$.
   At time 1:

$$\pi_1 = \Delta_0 S_1 + (\pi_0 - \Delta_0 S_0).$$

   Plug in u and d gives

$$\begin{aligned} 22/3 &= \Delta_0 16 + (\pi_0 - \Delta_0 8) \\ 0 &= \Delta_0 4 + (\pi_0 - \Delta_0 8) \end{aligned}$$

   Solving gives $\Delta_0 = 11/18$ and $\pi_0 = V_0 = 22/9$.

   (b) (10 points) Use the replicating portfolio approach to find the price V_0 of a European put option with strike 20 and expiration $N = 2$ on this stock.

Ans:

$$\begin{aligned} V_2(uu) &= 0 \\ V_2(ud) &= 12 \\ V_2(dd) &= 18. \end{aligned}$$

We have

$$\pi_2 = \Delta_1 S_2 + (\pi_1 - \Delta_1 S_1).$$

Plug in uu and ud in the above equation gives

$$\begin{aligned} 0 &= \Delta_1(u)32 + \pi_1(u) - \Delta_1(u)16 \\ 12 &= \Delta_1(u)8 + \pi_1(u) - \Delta_1(u)16. \end{aligned}$$

Solving gives $\Delta_1(u) = -1/2$ and $V_1(u) = \pi_1(u) = 8$.
Plug in ud and dd in the above equation gives

$$\begin{aligned} 12 &= \Delta_1(d)8 + \pi_1(d) - \Delta_1(d)4 \\ 18 &= \Delta_1(d)2 + \pi_1(d) - \Delta_1(d)4. \end{aligned}$$

Solving gives $\Delta_1(d) = -1$ and $\pi_1(d) = V_1(d) = 16$.
At time 1:

$$\pi_1 = \Delta_0 S_1 + (\pi_0 - \Delta_0 S_0).$$

Plug in u and d gives

$$\begin{aligned} 8 &= \Delta_0 16 + (\pi_0 - \Delta_0 8) \\ 16 &= \Delta_0 4 + (\pi_0 - \Delta_0 8) \end{aligned}$$

Solving gives $\Delta_0 = -2/3$ and $\pi_0 = V_0 = 40/3$.

■

6. Besides enrolling as a financial math student, Mr. Solve-alot also moonlights as a rock artist. His first single, titled "I like Big Math" has just debuted. He plans to sell the single for 8 dollars during the first week. During the second week, he'll either double or half the price of the first week, depending on the response. During the third week, he'll either double or half the price of the second week, again depending on the response. Mathematically we model the price of his single as followed:

$$\begin{aligned} S_1 &= 8 \\ S_2 &= 8X_1 \\ S_3 &= 8X_1X_2, \end{aligned}$$

where X_1, X_2 are i.i.d with distribution $P(X_i = 2) = q, P(X_i = 1/2) = 1 - q$. Assume that the interest rate $r = 0$.

(a) (5 points) Remembering his 485 lessons, Mr. Solve-alot wants to look at his price movement in a risk neutral way. He also recalls that the risk neutral measure in this case is the q so that $E^Q(X_1) = 1$. Determine this q.
Ans: $q = 1/3, 1 - q = 2/3$.

(b) (5 points) What is the risk neutral probability that during the 3rd week, the single will sell for more than or equal to 8 dollars?
Ans:

$$P(S_3 < 8) = P(S_3 = 2) = P(X_1 = X_2 = 1/2) = \frac{4}{9}.$$

So $P(S_3 \geq 8) = 5/9$.

(c) (5 points) Find an expression for $E^Q((K - S_3)^+)$, for a general constant K.

Ans:

$$E^Q((K - S_3)^+) = (K - 32)^+ 1/9 + (K - 8)^+ 4/9 + (K - 2)^+ 4/9.$$

(d) (5 points) Being a business savvy person (and not having great faith on his musical ability), Mr. Solve-alot wants to insure against the case that his single will fall below 10 dollars during the 3rd week. Can you advise him on what financial product he should buy, and help him price this financial product? (Feel free to use the result in part c without further explanation).

He should purchase a put option on his single, with strike 10 and expiration $N = 2$. The price of this is, by part 2

$$2(4/9) + 8(4/9) = 40/9.$$