Fundamental theorems of asset pricing - Continuous models

Math 485

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1 Introduction

In this note we discuss the two fundamental theorems of asset pricing for the continuous time model, in particular the Black-Scholes model. The mathematical tool for discussion is martingale theory in continuous time. We will define the notion of martingales in continuous time, and show that under the risk neutral measure, the discounted underlying asset price is a martingale. By our pricing formula, the discounted value process of a non-American financial derivative is also a martingale under the risk neutral measure. This is used to prove the first fundamental theorem of asset pricing. Under the uniqueness of the risk neutral measure, we show the existence of the hedging portfolio for non-American financial derivatives.

2 Martingale in continuous time

A process V_t is a martingale with respect to the filtration $\mathcal{F}_t^S$ under a probability measure P if:

a. $V_t \in \mathcal{F}_t^S$ for all t.

b. For all $t \geq s$, $E(V_t|\mathcal{F}_s^S) = V_s$.

Remark:

1. Condition (a) means that for each t, there exists a function f and a countable subset $\{t_0, t_1, t_2, \cdots\}$ of $[0, t]$ such that

$$V_t = f(S_{t_0}, S_{t_1}, \cdots).$$

This is consistent with our intuition that the value of the financial derivative should only depend on the historical price of the underlying asset.

2. Condition b is the martingale condition. The expectation E is taken under the probability P. This is essential: if we change the measure P, this condition may not hold. Thus V_t can be that a process is a martingale under some measure but not under some other measure.

Similarly, V_t is a sub (super)-martingale with respect to the filtration $\mathcal{F}_t^S$ under a probability measure P if:

a. $V_t \in \mathcal{F}_t^S$ for all t.

b. For all $t \geq s$, $E(V_t|\mathcal{F}_s^S) \geq (\leq) V_s$.

3. Sometimes we just say S_t is a martingale (under probability P). Then it is understood that the filtration is S_t's own filtration ($\mathcal{F}_t^S$).

2.1 Some examples

The following are the most important examples we encountered so far:

1. The Brownian motion is a martingale with respect to its own filtration.

2. The discounted stock price $e^{-rt}S_t$ is a martingale with respect to $\mathcal{F}_t^S$ under the risk neutral measure Q (but not neccessarily under the physical measure P).

3. The discounted value of a European option $e^{-rt}V_t$ is a martingale with respect to $\mathcal{F}_t^S$ under the risk neutral measure Q.

4. If $X_1, X_2, \cdots, X_n$ are martingales then $\sum_i c_i X_i$ is also a martingale, where c_i's are constants.

3 The first fundamental theorem of asset pricing

3.1 Betting against a martingale in continuous time

Recall the following situation in discrete time: Let $0 = t_0 < t_1 < ... < t_n = T$ be a partition of $[0, t]$. Let Δ_{t_k} be the number of shares we hold of S at time t_k. Our net "winning" over the period $[t_k, t_{k+1}]$ is $\Delta_{t_k}(S_{t_{k+1}} - S_{t_k})$ and our total winning up to a time t_i is

$$\pi_{t_i} = \sum_{k=0}^{i-1} \Delta_{t_k}(S_{t_{k+1}} - S_{t_k}).$$

We proved the following lemma:

Lemma 3.1. *If $\Delta_{t_k} \in \mathcal{F}_{t_k}^S$ and S_{t_k} is a martingale then π_{t_k} is also a martingale with respect to $\mathcal{F}_{t_k}^S$.*

Now suppose each period $[t_k, t_{k+1}]$ has length Δ_T and we let $\Delta_T \to 0$. If $t_i = t$ then

$$\sum_{k=0}^{i-1} \Delta_{t_k}(S_{t_{k+1}} - S_{t_k}) \to \int_0^t \Delta_u dS_u.$$

It can be shown also that the martingale property of π_{t_k} is preserved in the limiting process. That is we have the following continuous version of the above lemma:

Lemma 3.2. *If $\Delta_t \in \mathcal{F}_t^S$ and S_t is a martingale then π_t is also a martingale with respect to $\mathcal{F}_t^S$.*

The basic message is the same: when you bet against a martingale, the total winning remains a martingale as long as the strategy is non-anticipatory.

3.2 The dynamics of the value of a self-financing portfolio

Recall the following derivation: Let $0 = t_0 < t_1 < ... < t_n = T$ be a partition of $[0, T]$. Consider an investor who invests in an underlying asset S and the saving account such that the portfolio is self-financing. Let $\pi_k = \pi_{t_k}$ be the value of the portfolio and Δ_k be the number of shares of S he holds at time k. Then

$$\pi_{k+1} = \Delta_k S_{k+1} + (1 + r\Delta T)(\pi_k - \Delta_k S_k).$$

Rearraning terms, we have

$$\begin{aligned}\pi_{k+1} &= \pi_k + \Delta_k(S_{k+1} - S_k) + r\Delta_T(\pi_k - \Delta_k S_k) \\ &= \pi_k + \Delta_k(S_{k+1} - S_k) + y_k r(t_{k+1} - t_k).\end{aligned}$$

By induction (applying the same derivation on $\pi_k, \pi_{k-1}, \cdots$) we have

$$\pi_{k+1} = \sum_{i=1}^{k} \Delta_i(S_{i+1} - S_i) + y_i r(t_{i+1} - t_i).$$

where y_k is the amount of cash we holds at time k. Thus if we consider $\Delta(t), y(t)$ as a function of t, letting $\|\Delta\| \to 0$ we get

$$\pi_t = \int_0^t \Delta_u dS_u + \int_0^t y_u r du.$$

Now , self-financing requiring that $\Delta(t) + y(t) = \pi(t)$, replacing $y_u = \pi_u - rS_u$, then we have

$$\pi_t = \int_0^t \Delta_u (dS_u - rS_u du) + \int_0^t \pi_u r du,$$

Equivalently, we can write

$$d\pi_t = r\pi_t dt + \Delta_t (dS_t - rS_t dt).$$

This is the dynamics of the value of a self-financing portfolio in continuous time.

What we are interested in, however, is the *discounted value* of the portfolio: $e^{-rt}\pi_t$. Then by Ito's formula

$$d(e^{-rt}\pi_t) = -re^{-rt}\pi_t + e^{-rt}d\pi_t = e^{-rt}\Delta_t(dS_t - rS_t dt)$$

But again, similarly we have

$$d(e^{-rt}S_t) = -re^{-rt}S_t + e^{-rt}dS_t.$$

Thus

$$d(e^{-rt}\pi_t) = \Delta_t d(e^{-rt}S_t). \quad (1)$$

This is the dynamics of *the discounted value* of a self-financing portfolio in continuous time.

Remark 3.3. *Self-financing portfolio as a martingale*

From the formula (1)*, we see that if the discounted stock price* $e^{-rt}S_t$ *is a martingale, then it follows from Lemma* (3.2) *that the discounted portfolio value, as long as it is self-financing, is also a martingale. This result will play an important role in the fundamental theorems of asset pricing.*

3.3 Market with more than 1 assets

The result about self-financing portfolio also holds in market with more than 1 asset $S^1, S^2, \cdots, S^m$. The dynamics of a self-financing portfolio with m risky assets and a money market account just generalizes to

$$d(e^{-rt}\pi_t) = \sum_{i=1}^m \Delta_t^i d(e^{-rt}S_t^i).$$

(You can check this from the generalized self-financing condition for m risky assets in discrete time:

$$\pi_{k+1} = \sum_i \Delta_k^i S_{k+1}^i + e^{r\Delta T}(\pi_k - \sum_i \Delta_k^i S_k^i).)$$

It is clear that as long as each discounted asset price $e^{-rt}S_t^i$ is a martingale, and the number of S^i shares we hold at time t, Δ_t^i is non anticipatory: $\Delta_t^i \in \mathcal{F}_t^S$ for some t where $\mathcal{F}_t^S = \mathcal{F}_t^{S^1} \vee \mathcal{F}_t^{S^2} \vee \cdots \vee \mathcal{F}_t^{S^m}$, the smallest filtration containing the filtration generated by S^i then the discounted portfolio value $e^{-rt}\pi_t$ is also a martingale with respect to $\mathcal{F}_t^S$.

3.4 Self-financing portfolio in the Black-Scholes model

Recall that in the Black-Scholes model, under the risk neutral measure Q

$$dS_t = rS_t dt + \sigma S_t dW_t.$$

That is

$$d(e^{-rt}S_t) = \sigma e^{-rt} S_t dW_t.$$

Or

$$e^{-rt}S_t = S_0 + \int_0^t \sigma e^{-ru} S_u dW_u.$$

Again, since W_u is a martingale, by Lemma (3.2), $e^{-rt}S_t$ is a martingale. Thus by Remark (3.3), in the Black-Scholes model, the discounted value of a self-financing portfolio is a martingale under the risk neutral measure Q.

For market with m risky assets, we can also model each of them as following the Black-Scholes model with different Brownian motions and different volitility coefficients:

$$dS_t^i = rS_t^i dt + \sum_{j=1}^d \sigma^{ij} S_t^i dW_t^j, \tag{2}$$

where $W^1, W^2, \cdots, W^d$ are independent Brownian motions. Note that in this way, S^i, S^j can be correlated.

We are now in the position to discuss the first fundamental theorem of asset pricing in the continuous time.

3.5 The first fundamental theorem of asset pricing

Theorem 3.4. *Let a market have m risky assets $S^1, S^2, \cdots, S^m$. Suppose a risk neutral measure Q exists, that is*

$$S_s^i = E^Q(e^{-r(t-s)} S_t^i | \mathcal{F}_s^S), i = 1, \cdots, m.$$

Suppose additionally that all derivatives that make payment V_T at time T satisfy

$$V_t = E^Q(e^{-r(T-t)} V_T | \mathcal{F}_t^S),$$

then there is no self-financing portfolio consisting of S^i, V and the money market account such that $\pi_0 = 0$ and $P(\pi_t \geq 0) = 1, P(\pi_t > 0) > 0$ for $0 \leq t \leq T$. That is the market is arbitrage-free.

Proof.

Since $e^{-rt} V_t$ is a martingale by the condiion $V_t = E^Q(e^{-r(T-t)} V_T | \mathcal{F}_t^S)$, we can treat V as a risky asset. It follows from our dicussion about self-financing portfolio that any self-financing portfolio consisting of S^i, V and the money market account must satisfy $e^{-rt}\pi_t$ is a martingale under the risk neutral measure.

Thus by the martingale condition:

$$0 = \pi_0 = E^Q(e^{-rt}\pi_t).$$

Now suppose $P(\pi_t \geq 0) = 1$ and $P(\pi_t > 0) > 0$. Then it must follow (from an elementary result in measure theory) that

$$E^Q(\pi_t) > 0$$

which leads to $E^Q(e^{-rt}\pi_t) > 0$. Thus we cannot find such portfolio. ∎

3.6 Replicating portfolio as a pricing tool

Theorem (3.4) already states the pricing we must follow for any financial derivative if we want our market to be arbitrage free, whether or not we can find a replicating portfolio for the derivatives. We learned in Lecture 2b that we can also price a financial derivative by the replicating portfolio, if it exists. These two methods should be consistent, that is they should give the same price. The following Lemma confirms this is the case.

Lemma 3.5. *Let a market have m risky assets $S^1, S^2, \cdots, S^m$. If a risk neutral measure Q exists, that is*

$$S_t^i = E^Q(e^{-r\Delta T} S_T^i | \mathcal{F}_t^S), i = 1, \cdots, m.$$

Consider a financial derivative V, whose replicating portfolio exists. That is at any time $0 \leq t \leq T$, we can find Δ_t^i shares of asset S^i and y_t dollars in cash such that

$$\pi_t = \sum_i \Delta_t^i S_t^i + y_t = V_t,$$

and the portfolio is self-financing:

$$d(e^{-rt}\pi_t) = \sum_{i=1}^m \Delta_t^i d(e^{-rt} S_t^i).$$

Then $V_t = E^Q(e^{-r(T-t)} V_T | \mathcal{F}_t^S), \forall t$.

Proof. By what we discussed above, $e^{-rt}\pi_t$ is a martingale. Therefore

$$V_t = \pi_t = E^Q(e^{-r(T-t)}\pi_T | \mathcal{F}_t^S) = E^Q(e^{-r(T-t)} V_T | \mathcal{F}_t^S).$$

4 The second fundamental theorem of asset pricing

Theorem 4.1. *Let a market have m risky assets $S^1, S^2, \cdots, S^m$ and suppose they follow the multi-dimensional Black-Scholes model* (2)*. If a risk neutral measure Q exists, that is*

$$S_t^i = E^Q(e^{-r(T-t)} S_T^i | \mathcal{F}_t^S), i = 1, \cdots, m.$$

and it is **unique***, then every financial derivative that pays V_T at time T can be replicated and the market is arbitrage-free.*

Proof. The replicating condition requires that we are able to find a self-financing portfolio such that $V_T(\omega) = \pi_T(\omega), \forall \omega$. That is, we need to find $\Delta_t^i, i = 1, 2, \cdots, m$ so that

$$\begin{aligned} d(e^{-rt}\pi_t) &= \sum_{i=1}^m \Delta_t^i d(e^{-rt} S_t^i) \\ \pi_T &= V_T. \end{aligned}$$

Observe that since we know the unique risk neutral measure exists, V_0 is uniquely determined:

$$V_0 = E^Q(e^{-rT}V_T),$$

by the first fundamental theorem of asset pricing above.

We will set $\pi_0 = V_0$ (we're free to determine how to construct π).

Note that

$$e^{-rT}V_T - V_0 = e^{-rT}\pi_T - \pi_0 = \int_0^T d(e^{-ru}\pi_u) \;\; = \;\; \int_0^T \sum_{i=1}^m \Delta_u^i d(e^{-ru}S_u^i).$$

That is

$$\begin{aligned} e^{-rT}V_T &= V_0 + \int_0^T d(e^{-ru}\pi_u) = V_0 + \int_0^T \sum_{i=1}^m \Delta_u^i d(e^{-ru}S_u^i) \\ &= E^Q(e^{-rT}V_T) + \sum_{i=1}^m \int_0^T \Delta_u^i d(e^{-ru}S_u^i). \end{aligned}$$

Since S^i follows the Black-Scholes model (2),

$$d(e^{-ru}S_u^i) = \sum_{j=1}^d \sigma^{ij} S_t^i dW_t^j.$$

That is

$$e^{-rT}V_T \;\; = \;\; E^Q(e^{-rT}V_T) + \sum_{i=1}^m \sum_{j=1}^d \int_0^T \Delta_u^i \sigma^{ij} S_t^i dW_t^j.$$

A compact way to write this is to use matrix notation:

$$e^{-rT}V_T \;\; = \;\; E^Q(e^{-rT}V_T) + \int_0^T \overline{\Delta}_t^T \Sigma d\overline{W}_t,$$

where

$$\begin{aligned} \overline{\Delta}_t^T &= [\Delta_t^1 S_t^1, \Delta_t^2 S_t^2, \cdots, \Delta_t^m S_t^m] \\ \Sigma_{ij} &= \sigma^{ij} \\ \overline{W}_t^T &= [W_t^1, W_t^2, \cdots, W_t^m]. \end{aligned}$$

So now we see that at the core of it, the replicating portfolio question can be rephrased as *whether we can write* $e^{-rT}V_T$ *as an Ito integral.* The answer to this is positive, as long as V_T is measurable with respect to $\mathcal{F}^{\overline{W}_T}$ (this is a technical description to say that V_T is a function of the paths of W_t^i and the paths can only be sampled at countably many time points). It is referred to as the Martingale Representation Theorem.

In our set up, this condition for V_T is satisfied because it is a fuction of $S_T^i, i = 1, \cdots, m$ and S_T^i is measurable with respect to $\mathcal{F}^{\overline{W}_T}$. We will mention the Martingale Representation Theorem, and describe how we can apply it to solve for our replicating portfolio.

Theorem 4.2. *Let ξ be mesurable with respect to $\mathcal{F}^{\overline{W}_T}$ such that it is also a square integrable random variable. Then there exists $Z_t^1, Z_t^2, \cdots, Z_t^d$ such that Z_t^i is measurable with respect to $\mathcal{F}^{\overline{W}_t}$ and*

$$\xi = \sum_{i=1}^{d} \int_0^T Z_t^i dW_t^i.$$

So now applying the theorem, we can find a vector $\overline{Z}_t$ so that

$$e^{-rT} V_T = E^Q(e^{-rT} V_T) + \int_0^T \overline{Z}_t^T d\overline{W}_t.$$

The last thing we need to do is to see how we can solve for $\overline{\Delta}_t$ given $\overline{Z}_t^T$.

Note that the equation we're solving for is

$$\overline{\Delta}_t \Sigma = \overline{Z}_t^T,$$

or equivalently

$$\Sigma^T \overline{\Delta}_t = \overline{Z}_t.$$

This is where we need to use the uniqueness of Q. For this we need to describe a bit more details about Girsanov theorem.

Recall that in the physical measure, the dynamics of the underlying assets is as followed: for $i = 1, \cdots, m$

$$\begin{aligned} dS_t^i &= \mu^i S_t^i dt + S_t^i \sum_{j=1}^{d} \sigma^{ij} dW_t^j \\ &= r S_t^i dt + S_t^i [(\mu^i - r) dt + \sum_{j=1}^{d} \sigma^{ij} dW_t^j]. \end{aligned}$$

Suppose we can find a vector $\overline{\mu^Q}$ such that

$$\Sigma\overline{\mu^Q} = \overline{\mu} - \mathbf{1}r, \tag{3}$$

where

$$(\overline{\mu} - \mathbf{1}r)^T = [\mu^1 - r, \mu^2 - r, \cdots, \mu^m - r].$$

Then the above dynamics of S^i can be written as

$$\begin{aligned} dS_t^i &= rS_t^i dt + S_t^i \sum_{j=1}^{d} \sigma^{ij}[\mu_i^Q dt + dW_t^j] \\ &= rS_t^i dt + S_t^i \sum_{j=1}^{d} \sigma^{ij} d(W^Q)_t^j, \end{aligned}$$

where

$$d(W^Q)_t^j = \mu_i^Q dt + dW_t^j, j = 1, \cdots, d$$

are independent BMs under the measure Q given by Girsanov theorem.

Now note that the hypothesis that we have a **unique** measure Q is equivalent to the condition that the equation (3)

$$\Sigma\overline{\mu^Q} = \overline{\mu} - \mathbf{1}r,$$

have a unique solution.

Recall that we proved the following Lemma in Lecture 4b:

Lemma 4.3. *Let A be a $m \times n$ matrix. Suppose that* **there exists** *a vector $b \in \mathbb{R}^m$ such that the equation $Ax = b$ has a unique solution. Then for* **any** *vector $c \in \mathbb{R}^n$, the equation $A^T x = c$ has a solution.*

Using this lemma, we deduce that under the uniqueness hypothesis of Q, the equation

$$\Sigma^T \overline{\Delta}_t = \overline{Z}_t$$

has a solution. Then our replicating portfolio is

$$\Delta_t^i = \frac{\overline{\Delta}_t^i}{S_t^i}.$$