Fundamental theorems of asset pricing - Discrete models

Math 485

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1 Introduction

In this note we discuss the two fundamental theorems of asset pricing. The mathematical tool for discussion is martingale theory in discrete time. We will define the notion of martingales, and show that under the risk neutral measure, the discounted asset price is a martingale. By our pricing formula, the discounted value process of a non-American financial derivative is also a martingale under the risk neutral measure. This is used to prove the first fundamental theorem of asset pricing. Under the uniqueness of the risk neutral measure, we show the existence of the hedging portfolio for non-American financial derivatives. Finally, we show the existence of the hedging portfolio for American put option, by characterizing it as a super-martingale under the risk neutral measure.

2 Martingale in discrete time

A process V_k is a martingale with respect to the filtration $\mathcal{F}_k^S$ under a probability measure P if:

a. $V_k \in \mathcal{F}_k^S$ for all k.

b. For all $n \geq m$, $E(V_n | \mathcal{F}_m^S) = V_m$.

Remark:

1. Condition a means that each V_k is a function of $S_0, S_1, \cdots, S_k$. This is consistent with our intuition that the value of the financial derivative should only depend on the historical price of the underlying asset. Observe that $S_{k+1} \notin \mathcal{F}_k^S$ in our binomial model.

2. Condition b is the martingale condition. The expectation E is taken under the probability P. This is essential: if we change the measure P, this condition may not hold. Thus V_k can be that a process is a martingale under some measure but not under some other measure (think about the risk neutral meassure for example).

Similarly, V_k is a sub (super)-martingale with respect to the filtration $\mathcal{F}_k^S$ under a probability measure P if:

a. $V_k \in \mathcal{F}_k^S$ for all k.

b. For all $n \geq m$, $E(V_n | \mathcal{F}_m^S) \geq (\leq) V_m$.

3. Sometimes we just say S_k is a martingale (under probability P). Then it is understood that the filtration is S_k's own filtration $(\mathcal{F}_k^S)$.

2.1 Some examples

The following are the most important examples we encountered so far:

1. The discounted stock price $e^{-rk\Delta_T} S_k$ is a martingale with respect to $\mathcal{F}_k^S$ under the risk neutral measure Q (but not neccessarily under the physical measure P).

2. The discounted value of a European option $e^{-rk\Delta_T} V_k$ is a martingale with respect to $\mathcal{F}_k^S$ under the risk neutral measure Q.

3. The discounted value of an American option $e^{-rk\Delta_T} V_k$ is a super-martingale with respect to $\mathcal{F}_k^S$ under the risk neutral measure Q.

3 The first fundamental theorem of asset pricing

3.1 Betting against a martingale

A martingale is essentially a model for a fair game. First note that if V_k is a martingale with respect to $\mathcal{F}_k^S$ under P then $E(V_k) = E(V_{k-1}) = \cdots = E(V_0)$. Thus if you treat V_k as your total earning when investing in S then its expected earning is a constant in time if V_k is a martingale. Actually something stronger is true: your expected earning based on S at any time in the future, conditioned on the information up to the current time: $E(V_n | \mathcal{F}_m^S)$, is the same as your current earning based on S: V_m. This is what we think of as fair.

We can also look at the reverse direction. If S_k is a martingale with respect to $\mathcal{F}_k$ then the total earning under any strategy you can form investing using S_k is also a martingale, as long as your strategy at time k only uses the information about S up to time k (this excludes insider trading for example). More specifically, let Δ_k be

the number of shares we hold of S at time k, then our net "winning" over the period $[k, k+1]$ is $\Delta_k(S_{k+1} - S_k)$ and our total winning up to a time n is

$$\pi_n = \sum_{k=0}^{n-1} \Delta_k(S_{k+1} - S_k).$$

The notation π is used specifically to represent the value of our portfolio of S at time n. We have the following lemma:

Lemma 3.1. *If $\Delta_k \in \mathcal{F}_k^S$ and S_k is also a martingale then π_n is also a martingale with respect to $\mathcal{F}_n^S$.*

(From now on, we'll refer to any process X_k that has the property $X_k \in \mathcal{F}_k^S$ as *being adapted to $\mathcal{F}_k^S$*.)

Proof. It suffices to show $E(\pi_{n+1}|\mathcal{F}_n^S) = \pi_n$ (Why?). It is equivalent to show

$$E(\pi_{n+1} - \pi_n|\mathcal{F}_n^S) = 0.$$

But note that

$$\pi_{n+1} - \pi_n = \Delta_n(S_{n+1} - S_n).$$

Thus

$$\begin{aligned} E(\pi_{n+1} - \pi_n|\mathcal{F}_n^S) &= E(\Delta_n(S_{n+1} - S_n)|\mathcal{F}_n^S) \\ &= \Delta_n E(S_{n+1} - S_n|\mathcal{F}_n^S) = 0. \end{aligned}$$

Similarly, we can show that if S_k is a sub (super) martingale then π_n is a sub (super) martingale under similar conditions. A sub (super) martingale represents a game that favors one particular side of the game, either the house or the player.

3.2 Self-financing portfolio as a martingale

Remark 3.2. *In our model, S_k is NOT a martingale, but $e^{-rk\Delta_T}S_k$ is. But one* **does not invest in a discounted stock price process in reality**. *What one does is investing in (possibly multiple) financial assets and a saving account. The corresponding result is that if the portfolio is* **self financing** *then its discoutned value process is also a martingale under the risk neutral measure. The following lemma states the result more precisely.*

Lemma 3.3. *Suppose at any time k we hold Δ_k shares of the asset S and y_k in the saving account. Suppose $e^{-rk\Delta_T}S_k$ is a martingale. If Δ_k is adapted to $\mathcal{F}_k^S$ and the self-financing condition holds:*

$$\pi_{k+1} = \Delta_{k+1}S_{k+1} + y_{k+1} = \Delta_k S_{k+1} + y_k e^{r\Delta T}. \tag{1}$$

or equivalently

$$\pi_{k+1} = \Delta_k S_{k+1} + e^{r\Delta T}(\pi_k - \Delta_k S_k). \tag{2}$$

then $e^{-rk\Delta_T}\pi_k$ is also a martingale under $\mathcal{F}_k^S$.

Proof. Suppose $e^{-rk\Delta_T}S_k$ is a martingale. It is enough to show

$$E^Q\left[e^{-r\Delta T}\pi_{k+1}|\mathcal{F}_k^S\right] = \pi_k.$$

From the self-financing condition, we have

$$
\begin{aligned}
E^Q\left[e^{-r\Delta T}\pi_{k+1}\Big|\mathcal{F}_k^S\right] &= E^Q\left[e^{-r\Delta T}\Delta_k S_{k+1} + (\pi_k - \Delta_k S_k)\Big|\mathcal{F}_k^S\right] \\
&= \Delta_k E^Q\left[e^{-r\Delta T}S_{k+1} - S_k|\mathcal{F}_k^S\right] + \pi_k = \pi_k.
\end{aligned}
$$

Remark 3.4. *We do NOT have a similar conclusion in the self-financing portfolio case when $e^{-rk\Delta_T}S_k$ is a super (sub) martingale, that is correspondingly $e^{-rk\Delta_T}\pi_k$ is also a super (sub) martingale. The reason is the sign of Δ_k matters in this case. If we short an asset that is a super martingale (that is it decreases on average), then we're likely to make money in the future (that is the portfolio will be a sub-martingale). But if we long an asset that is a super martingale, then we're likely to lose money (that is the portfolio remains a super-martingale). The following calculation makes it clear:*

Suppose $e^{-k\Delta_T}S_k$ is a super martingale. Then

$$E^Q\left[e^{-r\Delta T}S_{k+1}|\mathcal{F}_k^S\right] \le S_k.$$

From the self-financing condition, we have

$$
\begin{aligned}
E^Q\left[e^{-r\Delta T}\pi_{k+1}\Big|\mathcal{F}_k^S\right] &= E^Q\left[e^{-r\Delta T}\Delta_k S_{k+1} + (\pi_k - \Delta_k S_k)\Big|\mathcal{F}_k^S\right] \\
&= \Delta_k E^Q\left[e^{-r\Delta T}S_{k+1} - S_k|\mathcal{F}_k^S\right] + \pi_k \\
&\le \pi_k, \ \text{if } \Delta_k \ge 0 \\
&\ge \pi_k, \ \text{if } \Delta_k \le 0.
\end{aligned}
$$

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3.3 Market with more than 1 assets

The result about self-financing portfolio also holds in market with more than 1 asset $S^1, S^2, \cdots, S^m$. We just have to generalize the self-financing condition to:

$$\pi_{k+1} = \sum_i \Delta_k^i S_{k+1}^i + e^{r\Delta T}(\pi_k - \sum_i \Delta_k^i S_k^i).$$

Our result is

Lemma 3.5. *Suppose at any time k we hold Δ_k^i shares of asset S^i and y_k in cash. Suppose $e^{-rk\Delta_T} S_k^i$ is a martingale for all i. If Δ_k^i is adapted to $\mathcal{F}_k^S$ and the self-financing condition holds:*

$$\pi_{k+1} = \sum_i \Delta_k^i S_{k+1}^i + e^{r\Delta T}(\pi_k - \sum_i \Delta_k^i S_k^i). \tag{3}$$

then $e^{-rk\Delta_T} \pi_k$ is also a martingale under $\mathcal{F}_k^S$.

Proof. Suppose $e^{-rk\Delta_T} S_k^i$ is a martingale for all i. It is enough to show

$$E^Q\left[e^{-r\Delta T}\pi_{k+1}|\mathcal{F}_k^S\right] = \pi_k.$$

From the self-financing condition, we have

$$
\begin{aligned}
E^Q\left[e^{-r\Delta T}\pi_{k+1}\Big|\mathcal{F}_k^S\right] &= E^Q\left[e^{-r\Delta T}\Big\{\sum_i \Delta_k^i S_{k+1}^i\Big\} + (\pi_k - \sum_i \Delta_k^i S_k^i)\Big|\mathcal{F}_k^S\right] \\
&= E^Q\left[\sum_i \Delta_k^i\Big\{e^{-r\Delta T} S_{k+1}^i - S_k^i\Big\}\Big|\mathcal{F}_k^S\right] + \pi_k \\
&= \sum_i \Delta_k^i E^Q\left[e^{-r\Delta T} S_{k+1}^i - S_k^i|\mathcal{F}_k^S\right] + \pi_k = \pi_k.
\end{aligned}
$$

3.4 The first fundamental theorem of asset pricing

We are in the position to prove the first version of the fundamental theorem of asset pricing for discrete time model.

Theorem 3.6. *Let a market have m risky assets $S^1, S^2, \cdots, S^m$. Suppose an equivalent risk neutral measure Q exists, that is Q is equivalent to P and*

$$S_k^i = E^Q(e^{-r\Delta T} S_{k+1}^i|\mathcal{F}_k^S), i = 1, \cdots, m.$$

Suppose additionally that all derivatives that make payment V_N at time N satisfy

$$V_k = E^Q(e^{-r\Delta T(N-k)} V_N|\mathcal{F}_k^S),$$

then there is no self-financing portfolio consisting of S^i, V and the saving account such that $\pi_0 = 0$ and $P(\pi_l \geq 0) = 1, P(\pi_l > 0) > 0$ for $0 < l \leq N$. That is the market is arbitrage free.

Proof.

Suppose at time 0 we hold Δ^i shares of asset S^i and y dollars in cash, as well as Δ shares of V such that $\pi_0 = 0$. Then because $e^{-rk\Delta T}V_k$ is also a martingale under Q, we conclude $e^{-rk\Delta T}\pi_k$ is a martingale by Lemma (3.5). Thus

$$E^Q\left(e^{-rl\Delta T}\pi_l\right) = \pi_0 = 0.$$

Now since we're in a discrete space model, there are only finitely many outcomes $\omega_1, \omega_2, \cdots, \omega_n$ at time l. Let $P^Q(\omega_i) = q_i$ and note that by the equivalence condition, $q_i > 0, \forall i$. Then

$$q_1\pi_l(\omega_1) + q_2\pi_l(\omega_2) + \cdots + q_n\pi_l(\omega_n) = 0.$$

Thus it must follow that either $\pi_l(\omega_i) = 0, \forall i$ or there exists i such that $\pi_l(\omega_i) < 0$.

Theorem 3.7. *Let a market have m risky assets $S^1, S^2, \cdots, S^m$. Suppose that there is no arbitrage opportunity in the market. Then a risk neutral measure Q exists, that is*

$$S_k^i = E^Q(e^{-r\Delta T}S_{k+1}^i|\mathcal{F}_k^S), i = 1, \cdots, m.$$

Moreover, all derivatives that make payment V_N at time N must satisfy

$$V_k = E^Q(e^{-r\Delta T(N-k)}V_N|\mathcal{F}_k^S),$$

Remark 3.8. *Theorem (3.6) and (3.7) together can be stated simply as a market is arbitrage free if and only if an equivalent risk neutral measure exists.*

Proof. WLOG we prove the statement for $N = 1$. We restate the statements of Theorem (3.6) and (3.7) into the following:

There exists a vector $Q = [q_1, q_2, \cdots, q_n]^T > 0$ so that

$$\begin{bmatrix} e^{-r\Delta T}S_1^1(\omega_1) & e^{-r\Delta T}S_1^1(\omega_2) & \cdots & e^{-r\Delta T}S_1^1(\omega_n) \\ e^{-r\Delta T}S_1^2(\omega_1) & e^{-r\Delta T}S_1^2(\omega_2) & \cdots & e^{-r\Delta T}S_1^2(\omega_n) \\ \cdots & & & \\ e^{-r\Delta T}S_1^m(\omega_1) & e^{-r\Delta T}S_1^m(\omega_2) & \cdots & e^{-r\Delta T}S_1^m(\omega_n) \\ 1 & 1 & \cdots & 1 \end{bmatrix} \begin{bmatrix} q_1 \\ q_2 \\ \cdots \\ q_n \end{bmatrix} = \begin{bmatrix} S_0^1 \\ S_0^2 \\ \cdots \\ S_0^m \\ 1 \end{bmatrix},$$

if and only if we cannot find a vector $\Delta = [\Delta_1, \Delta_2, \cdots, \Delta_m, \Delta_{m+1}]^T$ so that

$$
\begin{bmatrix} S_0^1 & S_0^2 & \cdots & S_0^m & 1 \end{bmatrix}
\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \cdots \\ \Delta_m \\ \Delta_{m+1} \end{bmatrix} = 0
\tag{4}
$$

and

$$
\begin{bmatrix}
S_1^1(\omega_1) & S_1^2(\omega_1) & \cdots & S_1^m(\omega_1) & e^{r\Delta T} \\
S_1^1(\omega_2) & S_1^2(\omega_2) & \cdots & S_1^m(\omega_2) & e^{r\Delta T} \\
\cdots \\
S_1^1(\omega_n) & S_1^2(\omega_n) & \cdots & S_1^m(\omega_n) & e^{r\Delta T}
\end{bmatrix}
\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \cdots \\ \Delta_m \\ \Delta_{m+1} \end{bmatrix} \gneqq 0,
\tag{5}
$$

where by $z \gneqq 0$ we mean $z \geq 0$ and $g \neq 0$.

Conditions (4) and (5) can be combined into one statement: we cannot find vector $\Delta = [\Delta_1, \Delta_2, \cdots, \Delta_m]^T$ so that

$$
\begin{bmatrix}
S_1^1(\omega_1) - e^{r\Delta T} S_0^1 & S_1^2(\omega_1) - e^{r\Delta T} S_0^2 & \cdots & S_1^m(\omega_1) - e^{r\Delta T} S_0^m \\
S_1^1(\omega_2) - e^{r\Delta T} S_0^1 & S_1^2(\omega_2) - e^{r\Delta T} S_0^2 & \cdots & S_1^m(\omega_2) - e^{r\Delta T} S_0^m \\
\cdots \\
S_1^1(\omega_n) - e^{r\Delta T} S_0^1 & S_1^2(\omega_n) - e^{r\Delta T} S_0^2 & \cdots & S_1^m(\omega_n) - e^{r\Delta T} S_0^m
\end{bmatrix}
\begin{bmatrix} \Delta_1 \\ \Delta_2 \\ \cdots \\ \Delta_m \end{bmatrix} \gneqq 0.
$$

Stated in this way, this is a well-known result in linear programming, known as Stiemke's Theorem. It is as followed: Let A be a $m \times n$ matrix. Then exactly one of the following system has a solution:

a. $y^T A \gneqq 0$ for some $y \in \mathbb{R}^m$

or

b. $Ax = 0, x > 0, x \in \mathbb{R}^n$.

Applying Stiemke's Theorem to our situation with

$$
A^T = \begin{bmatrix}
S_1^1(\omega_1) - e^{r\Delta T} S_0^1 & S_1^2(\omega_1) - e^{r\Delta T} S_0^2 & \cdots & S_1^m(\omega_1) - e^{r\Delta T} S_0^m \\
S_1^1(\omega_2) - e^{r\Delta T} S_0^1 & S_1^2(\omega_2) - e^{r\Delta T} S_0^2 & \cdots & S_1^m(\omega_2) - e^{r\Delta T} S_0^m \\
\cdots \\
S_1^1(\omega_n) - e^{r\Delta T} S_0^1 & S_1^2(\omega_n) - e^{r\Delta T} S_0^2 & \cdots & S_1^m(\omega_n) - e^{r\Delta T} S_0^m
\end{bmatrix},
$$

we see that if $y^T A \gneqq 0$ does not have a solution (the market is arbitrage free) then we must be able to find $x > 0 \in \mathbb{R}^n$ so that $Ax = 0$. You can easily see that this means we can find a probability vector Q (by normalizing x) so that for all $j = 1, \cdots, m$

$$\sum_i q_i(S^j(\omega_i) - e^{r\Delta T} S_0^j) = 0.$$

But then Q is exactly the risk neutral measure. Conversely, if a risk neutral measure exists, then the system $Ax = 0$ has a positive solution. Thus we cannot solve the system $y^T A \gneqq 0$. That is there is no arbitrage opportunity in the market.

Remark 3.9. *In Theorem* (3.6), *we did not include the American option in the portfolio. The reason is the discounted value of an American option is a super-martingale in general and the direction of the discounted portfolio value is unclear, as explained in Remark* (3.4). *However, one should also expect that the inclusion of American options should not affect the arbitrage property of the market. This is indeed the case, if the option holder acts in an optimal way. The description of this situation is slightly more complicated, so we reserve a separate section to discuss it.*

3.5   Hedging portfolio as a pricing tool

Theorem (3.6) already states the pricing we must follow for any financial derivative if we want our market to be arbitrage free, whether or not we can find a replicating portfolio for the derivatives. We learned in Lecture 2b that we can also price a financial derivative by the replicating portfolio, if it exists. These two methods should be consistent, that is they should give the same price. The following Lemma confirms this is the case.

Lemma 3.10. *Let a market have m risky assets* $S^1, S^2, \cdots, S^m$. *If a risk neutral measure Q exists, that is*

$$S_k^i = E^Q(e^{-r\Delta T} S_{k+1}^i | \mathcal{F}_k^S), i = 1, \cdots, m.$$

Consider a financial derivative V, whose replicating portfolio exists. That is at any time $0 \le k \le N$, we can find Δ_k^i shares of asset S^i and y_k dollars in cash such that

$$\pi_k = \sum_i \Delta_k^i S_k^i + y_k = V_k,$$

and the self-financing condition is satisfied:

$$\pi_{k+1} = \sum_i \Delta_k^i S_{k+1}^i + e^{r\Delta T}(\pi_k - \sum_i \Delta_k^i S_k^i).$$

Then $V_k = E^Q(e^{-r\Delta T(N-k)}V_N|\mathcal{F}_k^S), \forall k$.

Proof. By Lemma (3.5), $e^{-rk\Delta T}\pi_k$ is a martingale. Therefore

$$V_k = \pi_k = E^Q(e^{-r(N-k)\Delta T}\pi_N|\mathcal{F}_k^S) = E^Q(e^{-r(N-k)\Delta T}V_N|\mathcal{F}_k^S).$$

4 The second fundamental theorem of asset pricing

Theorem 4.1. *Let a market have m risky assets $S^1, S^2, \cdots, S^m$. If a risk neutral measure Q exists, that is*

$$S_k^i = E^Q(e^{-r\Delta T}S_{k+1}^i|\mathcal{F}_k^S), i = 1, \cdots, m.$$

*and it is **unique**, then every financial derivative that pays V_N at time N can be replicated and the market is arbitrage-free.*

We will first prove this theorem for the case $N = 1$ and then for general N. Because we're in a discrete space, there are n possible outcomes, $\omega_1, \omega_2, \cdots, \omega_n$ at time $N = 1$. The replicating condition requires that we are able to find $\Delta^i, i = 1, \cdots, m$ and y such that

$$\sum_{i=1}^m \Delta^i S_1^i(\omega_j) + ye^{r\Delta T} = V_1(\omega_j), j = 1, \cdots, n.$$

Note that the above is a system of n equations in $m+1$ variables. The unique martingale measure condition says that there exists a **unique** positive solution $q_1, q_2, \cdots, q_n$ to the system of equations

$$\sum_{i=1}^n q_i e^{-r\Delta T} S_1^j(\omega_i) = S_0^j, j = 1, \cdots, m$$

$$\sum_i q_i = 1.$$

Note that the above is a system of $m+1$ equations in n variables (the variables are the q_i). The LHS matrix in the first system is

$$
\begin{bmatrix}
S_1^1(\omega_1) & S_1^2(\omega_1) & \cdots & S_1^m(\omega_1) & e^{rT} \\
S_1^1(\omega_2) & S_1^2(\omega_2) & \cdots & S_1^m(\omega_2) & e^{rT} \\
\cdots \\
S_1^1(\omega_n) & S_1^2(\omega_n) & \cdots & S_1^m(\omega_n) & e^{rT}
\end{bmatrix}.
$$

It is equivalent to

$$
A = \begin{bmatrix}
e^{-r\Delta T}S_1^1(\omega_1) & e^{-r\Delta T}S_1^2(\omega_1) & \cdots & e^{-r\Delta T}S_1^m(\omega_1) & 1 \\
e^{-r\Delta T}S_1^1(\omega_2) & e^{-r\Delta T}S_1^2(\omega_2) & \cdots & e^{-r\Delta T}S_1^m(\omega_2) & 1 \\
\cdots \\
e^{-r\Delta T}S_1^1(\omega_n) & e^{-r\Delta T}S_1^2(\omega_n) & \cdots & e^{-r\Delta T}S_1^m(\omega_n) & 1
\end{bmatrix},
$$

as far as existence of solution is concerned.

The LHS matrix in the second system is

$$
B = \begin{bmatrix}
e^{-r\Delta T}S_1^1(\omega_1) & e^{-r\Delta T}S_1^1(\omega_2) & \cdots & e^{-r\Delta T}S_1^1(\omega_n) \\
e^{-r\Delta T}S_1^2(\omega_1) & e^{-r\Delta T}S_1^2(\omega_2) & \cdots & e^{-r\Delta T}S_1^2(\omega_n) \\
\cdots \\
e^{-r\Delta T}S_1^m(\omega_1) & e^{-r\Delta T}S_1^m(\omega_2) & \cdots & e^{-r\Delta T}S_1^m(\omega_n) \\
1 & 1 & \cdots & 1
\end{bmatrix}.
$$

Note that $A = B^T$. Thus what we need to prove is the following Lemma

Lemma 4.2. *Let A be a $m \times n$ matrix. Suppose that* **there exists** *a vector $b \in \mathbb{R}^m$ such that the equation $Ax = b$ has a unique solution. Then for* **any** *vector $c \in \mathbb{R}^n$, the equation $A^T x = c$ has a solution.*

This is a well-known result in linear algebra. We provide the proof for completeness.

Proof. If the equation $Ax = b$ has a unique solution then the equation $Ax = 0$ has a unique solution (and vice versa). The equation $Ax = 0$ has a unique solution if and only if the columns of A are linearly independent. But the columns of A are the rows of A^T. Thus the matrix A^T has full row rank. Thus for any vector c, the equation $A^T x = c$ has a solution.

Remark 4.3. *The fact that the system $Bq = S_0$ has a positive solution* **was not used** *in the Lemma. In fact it is not needed. We have seen this in the example of market*

that is complete but not arbitrage free. The condition for $q_i > 0$ is used to assert that the market is arbitrage free, and provide a link between pricing using expectation under the risk neutral measure and using the replicating portfolio, as described in Lemma (3.10).