# The multi-period binomial model (Cont)

Math 485

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# 1 Conditional expectation in the multi-period model

## 1.1 The value of a forward contract in the future

Suppose we're in the multi-period model with the present being $k = 0$. Consider the forward contract, which allows the holder to pay $K$ dollars for 1 share of the asset $S$ at time $N$. We've already discussed that its price $V_0$ should be $S_0 - Ke^{-rN\Delta T}$.

Now suppose at a time $n : 0 < n < N$ we want to sell this contract. How much should we charge it by? You should easily see that its price at time $n$ would be $V_n = S_n - Ke^{-r(N-n)\Delta T}$, by a replicating portfolio argument. But suppose we would like to apply the probabilistic approach in this case, how can we do it? Up to now, we used expectation under the risk neutral measure as a method for obtaining the no arbitrage price. But it's clear that taking expectation will not yield $V_n$ of the above forward contract; because taking expectation gives *a constant value*, while $V_n$ is clearly *a random variable.*

Of course the probabilistic approach can still be used, but instead of taking expectation we need to *take conditional expectation.* The intuition is that we are discussing a situation in the future, where *conditioning on the price of* $S_n$, we can decide the value of $V_n$. Indeed conditional expectation is fundamental in studying the multi-period model, as well as the continuous model later on. It is useful, for example, when we want to talk about not only the current price of a financial product, but its price evolution from time 0 to the expiration time $N$. We'll give a few examples for the multi-period model in the next section.

## 1.2 The flow of information

We mentioned that at time $n$, the value of $S_n$ is known to us. This is correct. But to be more precise, at time $n$, all values $S_0, S_1, \cdots, S_n$ are known to us. Thus in deciding the price of a financial product at time $n$, we need to condition on information of $S_0, S_1, \cdots, S_n$ instead of just $S_n$. This would be clear, for example, when we deal with path-dependent or exotic option.

We will then look at expressions of the form

$$E(f(S_{n+k})|S_0, S_1, \cdots, S_n), k \geq 0.$$

We introduce a notation that represents the amount of information regarding $S_k, k = 1, 2, \cdots, n$ available at time $n$ :$\mathcal{F}_n^S$. When the asset in mind is clear (i.e. we're only discussing 1 asset $S$), we'll drop the super-script $S$ and just write $\mathcal{F}_n$. Thus

$$E(f(S_{n+k})|S_0, S_1, \cdots, S_n) = E(f(S_{n+k})|\mathcal{F}_n^S) = E(f(S_{n+k})|\mathcal{F}_n).$$

Now because the process $S_k$ is Markov, we have

$$E(f(S_{n+k})|\mathcal{F}_n) = E(f(S_{n+k})|S_n).$$

Thus most of the time, conditioning on $S_n$ is sufficient. There are exceptions, for example, when we deal with path-dependent option. It is clear that

$$E(S_1 S_2|\mathcal{F}_2^S) = S_1 S_2 \neq E(S_1 S_2|S_2),$$

because

$$E(S_1 S_2|S_2) = S_2 E(S_1|S_2),$$

and generally $E(S_1|S_2) \neq S_1$.

## 1.3 Examples

When taking conditional expectation in the multi-period model, you should try to take advantage of the following:

1. The form of $S_n$ : $S_n = S_0 X_1 X_2 \cdots X_n$. 2. The i.i.d property of $X_i, i = 1, \cdots n$. 3. The elementary properties of conditional expectation. 4. The form of $f$ in $E(f_{S_{n+k}}|S_k)$.

**Example 1.1.**

$$E(S_4|S_2) = E(S_2 X_3 X_4|S_2) = S_2 E(X_3 X_4|S_2) = S_2 E(X_3)E(X_4) = S_2(pu + (1-p)d)^2.$$

**Example 1.2.**

$$E(S_3^2|S_2) = E((S_2X_3)^2|S_2) = S_2^E(X_3^2) = S_2(pu^2 + (1-p)d^2).$$

### 1.4 Conditional expectation revisited

When dealing with path-dependent options, we cannot rely on the Markovian property of $S$ as remarked above. So the following rule (the so-called tower property of conditional expectation) is important:

**If** $m \leq n$ then for any random variable $\xi$:

$$E(E(\xi|\mathcal{F}_n^S)|\mathcal{F}_m^S) = E(E(\xi|\mathcal{F}_m^S)|\mathcal{F}_n^S) = E(\xi|\mathcal{F}_m^S).$$

In other words, when you condition on more information, and then condition on less information, (or the other way) the result is always the same as conditioning on less information.

*Proof.* We prove

$$E(E(\xi|\mathcal{F}_n^S)|\mathcal{F}_m^S) = E(\xi|\mathcal{F}_m^S)$$

and leave the other equality as exercise. First note that $E(E(\xi|\mathcal{F}_n^S)|\mathcal{F}_m^S)$ is a function of $S_0, S_1, \cdots, S_m$ by definition. Let's call it $g(S_0, S_1, \cdots, S_m)$. We need to check for any function $f(S_0, S_1, \cdots, S_m)$

$$E\Big[g(S_0, S_1, \cdots, S_m)f(S_0, S_1, \cdots, S_m)\Big] = E\Big[\xi f(S_0, S_1, \cdots, S_m)\Big].$$

But by definition,

$$E\Big[g(S_0, S_1, \cdots, S_m)f(S_0, S_1, \cdots, S_m)\Big] = E\Big[E(\xi|\mathcal{F}_n^S)f(S_0, S_1, \cdots, S_m)\Big].$$

Observe that $f(S_0, S_1, \cdots, S_m)$ is also a function of $S_0, S_1, \cdots, S_n$ **since** $m \leq n$. Therefore,

$$E\Big[E(\xi|\mathcal{F}_n^S)f(S_0, S_1, \cdots, S_m)\Big] = E\Big[\xi f(S_0, S_1, \cdots, S_m)\Big].$$

## 2 The risk neutral measure

### 2.1 Motivation

In the multi-period model, we do not have to limit ourselves to only consider expiration time $n = N$. Consider a forward contract on the asset $S$ with 0 strike price that has expiration time $n \leq N$. What is the price for this contract at time $k$? Again, using the replicating portfolio apprach, you'll see that the price is $S_k$.

Recall how we define the risk neutral measure in the 1 period model as the measure $Q$ such that

$$E^Q(e^{-rT}S_T) = S_0.$$

The motivation for us there is exactly because the forward contract with 0 strike price expiration $T$ must be worth $S_0$ at time 0. Thus together with the above analysis, you can see that the the risk neutral measure $Q$ in the multi-period binomial model is such that for any $k \leq n$

$$E^Q(e^{-(n-k)\Delta T}S_n|S_k) = S_k. \tag{1}$$

## 2.2 The formula for the risk neutral measure

The equation (1) defines the risk neutral measure. But we want to find out concretely how to implement the risk neutral measure on the multi-period model, just as we did in the 1-period model. One important observation will help us here, that is *when limit to a 1 step period, such as from $n-1$ to $n$, the multi-period model looks exactly as a 1 period model. And the entire multi-period model can be re-produced by repeating so many such 1 step period movements.*

In terms of mathematics, what we're utilizing is the identical property of $X_i$. That is if we find out the distribution of $X_1$ under the risk neutral measure $Q$, then we've found out the distribution of all the $X_i$'s under $Q$ as well. And that completes the decription of risk neutral measure]

Concretely, the equation (1) for $n = 1$ and $k = 0$ reads

$$E^Q(e^{-\Delta T}S_1) = S_0.$$

But we have solved this equation before, of course. We conclude that $Q(X_1 = u) = q$ and $Q(X_1 = d) = 1 - q$ where

$$q = \frac{e^{r\Delta T} - d}{u - d}. \tag{2}$$

And thus under $Q$, $P(X_i = u) = q$ and $P(X_i = d) = 1 - q$ for all $i = 1, 2, \cdots, N$.

You may be suspicious. We derived this distribution from a 1 period analysis. Are we sure that the equation (1) holds for general $n$ and $k$?

To check, note this simple but also important observation:

$$E^Q(X_1) = \frac{e^{r\Delta T} - d}{u-d} u + \frac{u - e^{r\Delta T}}{u-d} d = e^{r\Delta T}.$$

Thus

$$\begin{aligned} E^Q(e^{-(n-k)\Delta T} S_n | S_k) &= E^Q(e^{-(n-k)\Delta T} S_k X_{k+1} X_{k+2} \cdots X_n | S_k) \\ &= e^{-(n-k)\Delta T} S_k [E(X_1)]^{n-k} = S_k, \end{aligned}$$

and equation (1) has been checked.

## 2.3 Pricing by risk neutral measure

**Theorem 2.1.** *Suppose the asset $S_n$ follows the multi-period binomial model, where the probability $S_n$ goes up is given by equation* (2)*. Then the no arbitrage price at time $k$ for any financial derivative with exercise time $N$ is*

$$V_k = E^Q(e^{-(N-k)\Delta T} V_N | \mathcal{F}_k^S). \tag{3}$$

*In particular, its value at 0 is*

$$V_0 = E^Q(e^{-N\Delta T} V_N).$$

Remark:

1. We will refer to equation (3) as the pricing formula (under risk neutral measure).

2. Note that in the pricing formula, the conditioning is on the *history of $S$, up to time $k$.* This formula becomes

$$V_k = E^Q(e^{-(N-k)\Delta T} V_N | S_k)$$

when we deal with Euro-style derivative for example. But in general, say, when dealing with exotic options, one cannot reduce conditioning on $\mathcal{F}_k^S$ down to $S_k$. Thus the pricing formula is a great theoeretical result for discussing the evolution of the derivative's price. Computing explicitly $V_k$ might take additional work.

3. The pricing formula also *only works for financial product with exercise time $N$.* In other words, it applies to Euro style and exotic derivatives, but NOT American option. We'll discuss why when we discusses the pricing of American options.

## 2.4 The fundamental theorems of asset pricing in multi-period model

You may also question the connection between the risk neutral measure, the existence of the replicating portfolio and the non-existence of arbitrage opportunity. Similar to the one period model, we also have two fundamental theorems that establish their connection here:

**Theorem 2.2.** *In the multi-period binomial model, the risk neutral measure exists if and only if there is no arbitrage opportunity.*

**Theorem 2.3.** *In the multi-period model, the risk neutral measure exists, and is unique, if and only if there is a replicating portfolio.*

Intuitively, these theorems are true because when we limit to any one step period, the multi-period model "looks like" the 1 period model. We have checked that for the one-period model, these theorems are true.

# 3 Remarks on using the binomial tree for pricing

It is common to use the "backward stepping" method to price a financial asset in the multi-period binomial model. This again makes use of the formula (3), where now we replace $N$ by $k+1$, by the property of conditional expectation:

$$V_k = E^Q(e^{-\Delta T} V_{k+1} | \mathcal{F}_k^S).$$

Even more explicitly, if we denote $\omega$ to be a vector of length $k$ consisting of $u$ and $d$ (so that $\omega$ denotes an outcome at time $k$) then the above formula becomes

$$V_k(\omega) = e^{-\Delta T}[qV_{k+1}(\omega u) + (1-q)V_{k+1}(\omega d)]. \tag{4}$$

This is reduced further, in the case of Euro-style options, to finding the value of $V_k$ at "a certain node" on the binomial tree. That is at time $k$ there are $k+1$ nodes then the price $V_k$ at a particular node $i, i = 1, \cdots, k+1$ can be computed as

$$V_k(i) = e^{-\Delta T}[qV_{k+1}(iu) + (1-q)V_{k+1}(id)].$$

Note that this amounts to saying for any outcomes $\omega_1, \omega_2$ of length $k$ that consists of the same portion of $u$ and $d$, we have

$$V_k(\omega_1) = V_k(\omega_2).$$

This is only valid, of course, if the financial asset is *Markovian*, because in that case $V_k$ only depends on $S_k$ (and not $S_{k-1}, S_{k-2}, \cdots$), and $S_k(\omega_1) = S_k(\omega_2)$.

However, this finding value at a "certain node" will no longer be valid in a path dependent option, for exapmple a down and out option. It is because now $V_k$ depends not only on $S_k$, but also on $S_{k-1}, S_{k-2}, \cdots$. So it could happen that $S_k(\omega_1) = S_k(\omega_2)$ but $S_{k-1}(\omega_1) \neq S_{k-1}(\omega_2)$ etc. So one cannot conclude that $V_k(\omega_1) = V_k(\omega_2)$. But the formula (4) is still valid. That is the option has to be priced via a "path by path" method.