The multi-period binomial model

Math 485

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1 Description of the multi-period binomial model

To make our model richer, we'll transition from the 1-period model to the multi-period binomial model. Specifically we'll have:

1.1 Notations

The present time, denoted at $n = 0$ and the expiration time, denoted at $n = N$. The lower case letter n, k (and occasionally i, j, m) will be used to denote the time variable. The notation S_k (or $S_n, S_i, S_j \cdots$) denotes the value of the stock (the underlying) at time k (or time $n, i, j \cdots$). An important convention we'll use is that S_0 *will always be a constant*, that is the present value of the stock is always known. For any $k \geq 1$, S_k is *a random variable*. The specific distribution of S_k will be discussed below.

Similarly, we'll denote V_k to be the value of a specific financial product at time k. In particular, if V is the European call option with strike K and expiration N, then $V_N = (S_N - K)^+$. We'll also denote π_k to be the value of a specific portfolio at time k. In particular, if the portfolio is *replicating* then $V_N = \pi_N$. Also note that V_0, π_0 are also constants, and for $k \geq 1$, V_k, π_k are *random variables*.

We will suppose that the time intervals between any two discrete moments $k, k+1$ are the same, denoted as ΔT. Thus the expiration time can also be written as $T = N \Delta T$.

For a replicating portfolio, we will denote the number of shares of S we hold at a particular time as Δ_k (do not confuse this with the interval length ΔT. In general, Δ_k will also be *a random variable* (which is easy to understand, as the number of shares we hold at time k will depend on the actual value of S_k at that time).

The interest rate will be denoted as r.

1.2 The evolution of the stock

At any time k, there are two possibilities for the stock to evolve: either jump up to $S_{k+1} = uS_k$ or jump down to $S_{k+1} = dS_k$ for some $d < e^{r\Delta T} < u$. Moreover, the distribution of S_{k+1} *only depends on S_k and not any further history of S.* In mathematical notation we write:

$$P(S_{k+1} = x | S_0, S_1, \cdots, S_k) = P(S_{k+1} = x | S_k),$$

for any real value x. That is, as far as deciding the behavior of S_{k+1}, the only information we need is the behavior of S_k. Any further information about its past history is irrelevant. We call this property of a random process (which $S_k, k = 1, \cdots N$ is) *the Markov property.* The theory of Markov chain is discussed in more details in the second semester of probability theory class 478.

An example of something that *is not* a Markov chain is any process whose evolution depends on more than its immediate history. For example, you *may* argue that the distribution of tomorrow's weather (whether it's gonna rain, shine, snow, be windy etc.) does not just depend on today's weather but also on the last couple of days. I.e. if it has been raining for the last 2 days then the chance of its continuing raining is higher than if it's only been raining today. In this sense, the day by day weather is not a Markov process. But one can consider the weather of a two or three days in a row, and then it may be a Markov process. This is a technique called changing the state space of a process. This particular discussion is only for your information. In this class we'll only be dealing with process that is already Markov to start with.

1.3 The probability space and the state space

You can easily see that for a given k, there are only $k + 1$ values S_k can take. Namely

$$S_k = S_0 u^i d^{k-i}, i = 0, 1, \cdots, k.$$

It is convenient to have notations to refer to this event. We'll give an example when $N = 3$.

It is clear that when $N = 3$, there are only 8 possible outcomes, namely

$$\Omega = \Big\{ uuu, uud, udu, duu, ddu, dud, udd, ddd \Big\},$$

which corresponds to, for example

$$S_3(uuu) = S_0 u^3, S_3(dud) = S_0 d^2 u, S_2(uud) = S_0 u^2, S_0(udu) = S_0 ud \cdots$$

From this list we can create other events, such as

$$\{uu\} = \{uuu\} \cup \{uud\}, \{ud\} = \{udu\} \cup \{udd\} \cdots$$

(and events at the time $k = 1$ level etc)

Note that these events are outcomes for S_2 but *not outcome* for S_3. Thus it makes sense to say $S_2(ud) = S_0 ud$ but not $S_3(ud)(=?)$.

We'll refer to Ω as our *probability space* and the values S_k can take as our *state space* for S_k. Note that for a particular N, there are 2^N outcomes in Ω, and there are $k + 1$ members in the state space of S_k.

1.4 A mathematical construct of S_k

We will construct our multi-period model for S_k out of *independent, identically distributed random variables that represent jump size.* This will be a very important idea, that you may see again in the construction of general Markov processes, or in Brownian motion, an important ingredient in our continous model later on.

More specifically, let $X_1, X_2, \cdots, X_N$ be i.i.d random variables with distribution

$$P(X_1 = u) = p; P(X_1 = d) = 1 - p,$$

for some $0 < p < 1$. (Note: since these random variables are *identical*, we only need to prescribe the distribution of X_1 and the rest would have the same distribution).

Then we simply define

$$S_k := S_0 X_1 X_2 \cdots X_k = S_0 \prod_{i=1}^{k} X_i.$$

Note that in this way we have a recursion relationship between S_i and $S_j, i < j$ which comes in handy when we compute conditional expectation (discussed later on in this lecture)

$$S_j = S_i X_{i+1} X_{i+2} \cdots X_j = S_i \prod_{k=i+1}^{j} X_k.$$

We claim that this way we recover the description of the evolution of S_k given above. Indeed, it is easy to see the probability space and state space are the same. The only thing to check is the Markovian property of S_k. We will give a rigorous justification

when when we discuss conditional expectation in the multi-period model. For now, let's just give some intuition why it is true. We have

$$P(S_{k+1} = x | S_1, S_2, \cdots, S_k) \quad = \quad P(S_k X_{k+1} = x | S_1, S_2, \cdots, S_k)$$

And you see that X_{k+1} is *independent of* $S_1, S_2, \cdots, S_k$ by construction. Thus conditioning on $S_1, S_2, \cdots S_k$ is the same as *not conditioning* as far as X_{k+1} is concerned (recall that if X is independent of Y then $E(X|Y) = E(X)$). On the other hand, the only information we need to determine S_k is S_k itself, and we don't need $S_1, S_2, \cdots, S_{k-1}$. Putting these two facts together, you can believe that

$$P(S_k X_{k+1} = x | S_1, S_2, \cdots, S_k) = P(S_k X_{k+1} | S_k),$$

which is the Markov property.

This description will be important when we use the probabilistic approach (or expectation approach) to pricing a financial product. One important remark here is that *the replicating portfolio approach still works in multi-period model.* The only drawback is it is computationally intensive (you'll see why). Thus if all we are interested in is pricing, then the expectation approach is more efficient. We'll treat the replicating portfolio approach in the in the final section of this note.

2 The distribution of S_k

We have the following result:

Lemma 2.1. *In the above construction, S_k has a binomial distribution. Namely*

$$P(S_k = S_0 u^i d^{k-i}) = \binom{k}{i} p^i (1-p)^{k-i}.$$

Remark: This is not the typical $Bin(k, p)$ distribution that you're used to, *as far as the values S_k takes is concerned.* But if you consider any time S goes up as a success, and count how many times it goes up until time k, then indeed we get a Binomial distribution in the traditional sense.

Proof. Recall that $S_k = S_0 X_1 X_2 \cdots X_k$. Clearly $S_k = S_0 u^i d^{k-i}$ if and only if i of $X's$ take value u and $k - i$ of them take value d. Because they are independent, the probability for a particular arrangement to happen is $p^i(1-p)^{k-i}$. There are $\binom{k}{i}$ such arrangements, since we just choose i of them among k total to take value u.

3 Financial products in multi-period model

The multiperiod model allows for a richer variety of financial products, namely the products that can depend on the *past history* of the stock. We'll describe the financial products we'll encounter in this course for the multi-period model below.

3.1 The Euro-style options

A financial product that makes a payment $f(S_N)$ for some deterministic function f at time N is referred to as a Euro-style option. For example, if $f(x) = (x - K)^+$ then we have the Euro-call and $f(x) = (K - x)^+$ then we have the Euro-put option. But we can also choose $f(x) = x^2, f(x) = \sin(x)$ or $f(x) = (x^2 - K)^+$. Of course the question about the financial interpretation of these products can be raised (what kind of advantage to the buyers do they offer?). But mathematically, we can treat all of these the same ways as we treat the Euro-put and Euro-call options. Moreover, these additional examples allow for more tractable mathemtical problems to be worked on. To recap, the distinguishing features of a Euro-style option is that the *exercise time is only at the expiration time* N and *the dependence of the payoff is only on the value of the stock at the expiration time S_N.*

3.2 Exotic options

Now that we're in multi-period model, it makes sense to talk about a history of S on the time interval $0, 1, \cdots, N$. A financial product that makes payment $f(S_0, S_1, \cdots, S_N)$ for some deterministic function f at time N is referred to as an exotic-option. Thus like a Euro-style option, the exercise time of an exotic option is still the expiration time N, but the pay off can be dependent on the past history of S.

Remark: You may argue whether the word *option* has any meaning in these contexts, since we do not have the interpretation of a choice here. This is correct, but simply the word option here can be understood as a financial product. It is also the terminology employed in the finanical literature, so we'll follow the convention here.

Some examples of exotic options: The **look back** option:

$$f(x_1, x_2, \cdots, x_N) = \max_{i=1, \cdots, N} x_i.$$

We call it a look back option since you get the *best* value of the stock in its past history as the payment.

On the other hand, if the payment is the average of the stock value in its history, then we get the **Asian** option:

$$f(x_1, x_2, \cdots, x_N) = \frac{1}{N} \sum_{i=1}^{N} x_i.$$

We also have the **barrier options**: if the stock ever get below or above some threshold, then the option is activated or the option becomes worthless. The first scenario is referred to as **down (or up) and in options**. The second is referred to as **down (or up) and out options.** The barrier options can in turn be call or put option. It means if the option *ever gets activated* (or *never knocked out*) then its payment at the expiration time will be the same as a Euro call or put option. In particular, let's consider a barrier call option with strike K and barrier L (L is a constant). The function $f(x_1, x_2, \cdots, x_N)$ in this case takes the following form:

Up and in:

$$f(x_1, x_2, \cdots, x_N) = \mathbf{1}_{\{\max_{i=1,\cdots,N} x_i \geq L\}} (x_N - K)^+.$$

Up and out:

$$f(x_1, x_2, \cdots, x_N) = \mathbf{1}_{\max_{\{i=1,\cdots,N} x_i \leq L\}} (x_N - K)^+.$$

Down and in:

$$f(x_1, x_2, \cdots, x_N) = \mathbf{1}_{\min_{\{i=1,\cdots,N} x_i \leq L\}} (x_N - K)^+.$$

Down and out:

$$f(x_1, x_2, \cdots, x_N) = \mathbf{1}_{\{\max_{i=1,\cdots,N} x_i \geq L\}} (x_N - K)^+.$$

Where, for a real number x, we have

$$\begin{aligned} \mathbf{1}_{\{x \geq L\}} &= 1 \text{ if } x \geq L \\ &= 0 \text{ if } x < L. \end{aligned}$$

3.3 American options

The last product we'll encounter is the American option, which is similar to a European option, with the additional feature that it gives the holder *the right to choose the time to exercise* the option with a fixed strike price K. More specifically, for

an American *call* option, the option holder can choose a time between 0 and N to exercise and *pay K* dollars for a share of S. For an American call option, the option holder can choose a time between 0 and N to exercise and *sell* a share of S for K dollars.

4 Pricing by the replicating portfolio approach

4.1 Self-financing portfolio

The idea of pricing using the replicating portfolio approach is the same: we would like to construct a portfolio, initially with Δ_0 shares of stock and y dollars in the money market such that at the expiration time

$$\Delta_N S_N + y_N = V_N, \tag{1}$$

where V_N is the value of the financial product and y_N is the amount of cash in the portfolio at time N. Notice that here we use Δ_N and y_N versus the construction in the 1 period model, where the number of shares of stocks remains the same and the money market account only becomes ye^{rT}. The reason is in the multi-period model, we can't expect to select our portfolio at time 0 and leave it unattended, hoping that equation (1) will hold at time N. Such approach is called the **buy and hold** approach for a portfolio; and obviously we can't generally use such approach for pricing in multi-period model. (One reason is you can visualize, with the buy and hold approach, you're treating the multi-period model as one big one-period model. However, now at the terminal time T, S_N can take $N+1$ values, in stead of just 2 values as before. Thus one cannot solve for Δ_0 and y in a over-determined system, as discussed in the previous lecture).

Thus we need to re-balance our portfolio at each time period $1, 2, \cdots, N$. How we select the portfolio will be addressed next; but first we need to note there is one obvious constraint, if the portfolio is replicating. That is the portfolio has to be *self-financing*, i.e. one cannot put additional funding into the portfolio, nor can one withdraw the cash from it. Letting y_k be the amount of cash in the portfolio at time k, the **self-financing condition** can be expressed as:

$$\pi_{k+1} = \Delta_{k+1} S_{k+1} + y_{k+1} = \Delta_k S_{k+1} + y_k e^{r\Delta T}. \tag{2}$$

The interpretation is this: we hold Δ_k shares of stock and y_k dollars at time k. At time $k+1$, the stock price changes to S_{k+1} and the money market grows to $y_k e^{r\Delta T}$.

This is our portfolio value at time $k+1$, π_{k+1}. Now we can rebalance our portfolio, if we want to. But all we at our disposal is π_{k+1} dollars. Thus if we want to buy Δ_{k+1} shares of stock at price S_{k+1}, then the amount of cash we have in the bank, y_{k+1} has to be such that

$$\pi_{k+1} = \Delta_{k+1} S_{k+1} + y_{k+1}.$$

Note that throughout this discussion, y_k can be negative, with the interpretation that we borrow money from the bank, in stead of putting it in a saving account. Note that since

$$\pi_k = \Delta_k S_k + y_k,$$

it follows from (2) that

$$\pi_{k+1} = \Delta_k S_{k+1} + e^{r\Delta T}(\pi_k - \Delta_k S_k). \tag{3}$$

This equation can be looked as as a recurrence relation between π_{k+1} and π_k. It has the advantage that the cash holding y_k and y_{k+1} do not show up in the equation, thus reducing the number of unknowns we have to deal with (only Δ_k needs to be found). Observe that both (2) and (3) are equivalent, thus we'll refer to either one as the self-financing condition (or equivalently, the *self-financing equations*). More often equation (3) will be used. However, you should still use your judgement to decide which equation is best for which situation.

The self-financing equation (3) will be *the key* to solve for the replicating portfolio. The pricing by the replicating portfolio approach in the 1-period model is simply an application of the self-financing condition, as we'll show in the next section.

4.2 Self-financing equation in 1 period model

In the 1 period model, $N = 1$ thus the equation (3) reduces to

$$\pi_1 = \Delta_0 S_1 + e^{r\Delta T}(\pi_0 - \Delta_0 S_0).$$

How to use this equation? The portfolio is replicating, thus $pi_1 = V_1$. Keeping in mind that this equation has to hold for all outcomes, we have

$$
\begin{aligned}
V_u &= V_1(u) = \Delta_0 S_1(u) + e^{r\Delta T}(\pi_0 - \Delta_0 S_0) = \Delta_0 u S_0 + e^{r\Delta T}(\pi_0 - \Delta_0 S_0) \\
V_d &= V_1(d) = \Delta_0 S_1(d) + e^{r\Delta T}(\pi_0 - \Delta_0 S_0) = \Delta_0 d S_0 + e^{r\Delta T}(\pi_0 - \Delta_0 S_0).
\end{aligned}
$$

Thus we have $\Delta_0 = \frac{V_u - V_d}{S_0(u-d)}$, exactly as we had before.

4.3   The steps of solving the self-financing equation

4.4   Non-path dependent option

A recursive approach is used to find the replicating portfolio using the self-financing equation. We start at time $k = N - 1$ where equation (3) reads as

$$\pi_N = \Delta_{N-1} S_N + e^{r\Delta T}(\pi_{N-1} - \Delta_{N-1} S_{N-1}).$$

What are we solving for? **We are solving for Δ_{N-1} and π_{N-1}.** Once we know these then it is clear that we known how to construct a hedging portfolio at time $N - 1$. However, observe that the above equation is an equation of *random variables*, so clearly we are NOT solving for explicit (numerical) value of Δ_{N-1} and π_{N-1}. Instead, we should ask we are solving for Δ_{N-1} and π_{N-1} **in terms of what variable**? The answer is we are solving for Δ_{N-1} and π_{N-1} in terms of S_{N-1}.

A crucial observation here is that since we are at time $k = N-1$, S_{N-1} *is a known value*. This has a real-life interpretation that once we arrive at time $N - 1$, then by observing the stock market, we know what S_{N-1} is. Since Δ_{N-1} and π_{N-1} are expressed in terms of S_{N-1}, we can balance our portfolio accordingly.

What about π_N, how do we know its value? This is where *the replication property is used*. Since the portfolio is replicating, $\pi_N = V_N$ and if we are dealing with say a Euro-style derivative, then we can replace V_N with $g(S_N)$ for some function g.

What about S_N? This is where *the binomial model is used*. Namely, for a given value of S_{N-1} (recall we assumed it's known, since we're at time $N - 1$), the only values S_N can take are uS_{N-1} or dS_{N-1}, thus essentially reducing us to the 1-period model's case. More details will be given in the example in class.

Now assume that we have solved this equation at time $N-1$. Proceed backwardly, the next equation at time $N - 2$ is

$$\pi_{N-1} = \Delta_{N-2} S_{N-1} + e^{r\Delta T}(\pi_{N-2} - \Delta_{N-2} S_{N-2}).$$

But now we're back to the same procedure above, with π_{N-1} being function of S_{N-1}, S_{N-2} is known, and we're solving for Δ_{N-2} and π_{N-2} in terms of S_{N-2}.

Thus you see that *provided, at each step k we can solve for Δ_k and π_k* then keep on going we will arrive at step $k = 0$, at which point we have completely obtained our replicating portfolio set up, as well as the price of the financial product V_0, given by π_0.

It is not hard to be convinced that we can solve for Δ_k and π_k at each step k. An abstract proof can be given, but it can be messy. We'll demonstrate this by an example with $N = 3$ in class and you'll see how one can generally prove this fact.

4.5 Path dependent option

If the derivative is path-dependent, then the above system of equations needs to be replaced with

$$g(S_N, S_{N-1}, \cdots, S_0) = \Delta_{N-1}S_N + e^{r\Delta T}(\pi_{N-1} - \Delta_{N-1}S_{N-1}).$$

So now, we are solving for π_{N-1} as a function of $S_{N-1}, S_{N-2}, \cdots, S_0$ instead of just S_{N-1}. This has to hold true for all $N + 1$ outcomes of S_N. The question is, of course, is the above system solvable? The answer is yes.

The key again is for any outcomes of $s_{0,1}, \cdots, s_{N-1}$, S_N can only take 2 values us_{N-1}, ds_{N-1}. Thus the above equation reads, for a particular outcome

$$
\begin{aligned}
g(us_{n-1}, s_{N-1}, \cdots, s_0) &= \Delta_{N-1}(s_0, s_1, \cdots, s_{N-1})us_{N-1} \\
&\quad + e^{r\Delta T}(\pi_{N-1}(s_0, s_1, \cdots, s_{N-1}) - \Delta_{N-1}s_{N-1}) \\
g(ds_{n-1}, s_{N-1}, \cdots, s_0) &= \Delta_{N-1}(s_0, s_1, \cdots, s_{N-1})ds_{N-1} \\
&\quad + e^{r\Delta T}(\pi_{N-1}(s_0, s_1, \cdots, s_{N-1}) - \Delta_{N-1}s_{N-1})
\end{aligned}
$$

This is a system of 2 equations with 2 unknowns, and you can verify that it is solvable.