Introduction to basic financial products and pricing

Math 485

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1 The financial products

1.1 Forward contract

Suppose you run a factory, and you know that in November you will need a large amount of oil (say 10,000 barrels). Suppose the price of oil now is 100 USD per barrel, but for certain reason you do not want to purchase 10,000 barrels right now (either because you did not have the cash available yet, or the cost of inventory will be very high from now until November when you actually need the oil). But waiting until November to purchase is also risky, since the price may jump to 115 USD per barrel, for example. So what you would want to do is to enter into a *contract*, to *lock down* a price for a *future purchase* of oil, at say, 105 USD per barrel (or another price of your choice). In this way, we say you have entered into a **forward contract** with **expiration date** on Nov 1, for a barrel of oil with **strike price** 105 USD. (For simplicity we suppose each contract is just for 1 barrel. To lock in the price for 10,000 barrels you simply buy 10,000 contracts). We will also refer to whatever the contract is based on (the oil in this case) as **the underlying**.

The question is, what is *the fair price* of such a contract? More importantly, what do we mean by *a fair price*?

1.2 Call option

One feature of the forward contract is that once you enter it, you *must* purchase the product on the expiration date. Now suppose it comes November and the oil price drops to 95 dollars per barrel. Then if you're already in a forward contract, you're in an undesirable situation. So you may want to enter into a more flexible

agreement than the forward contract, one that allows you to have **an option** to purchase, depending on how the situation turns out. This is what a call option is for. We say a **call option** on a certain product with **expiration date** T and **strike price** K, is a contract that gives you the **right, but NOT the obligation**, to purchase one unit of the product at time T and with price K.

Remark: You don't have to be in the financial world or run a business to encounter call option. United Airlines has a feature called Farelock, where with a small fee, you can lock in the price for a ticket for 1 day, 2 days or a week. If you don't exercise your "right" to buy the ticket before the deadline, you simply lose that fee. This is exactly a call option on an airplane ticket.

Again, the question is what is *the fair price* for such a call option? If you ever purchase a Farelock with United Airlines, you'll note that the price is about 8 dollars for a fare lock of a ticket of value of about 400 dollars for a week. Is this a fair price?

1.3 Put option

So far, we've discussed examples on a buyer's side. Now suppose you're in the sales business and you have some product to sell, say soy. You won't harvest soil until say October, but you would want to lock in a price as well. If you just want to lock in a price, then a forward contract is what you want. But if you want to lock in a price, and also want the flexibility to sell if the price is higher when it comes to the expiration date, then you want to enter into a **put option**. Put simply, a **put option** is just the exact oppposite of a call option. That is, a **put option** on a certain product with **expiration date** T and **strike price** K, is a contract that gives you the **right, but NOT the obligation**, to **sell** one unit of the product at time T and with price K.

1.4 Other products

There are many other options, for example the American option, or other *exotic options* such as the Look back, Asian, Bermudan, Barrier options etc. We will discuss these later in this class. Note that more precisely, the two options we discuss above are called the **European options**. There are many many other financial products besides these. In this class, our focus is to study the main tools and the main models people use for pricing the more "elementary" products listed here.

2 Different approaches to pricing

When you have a product (be it a tangible product, like a laptop, or a service, like a legal consulting, or a random game, like a casino game), you would like to know what its fair price is. Here are some possible approaches you may take:

2.1 By cost of components

You may decide how much the components of your product cost, and the price is simply the sum of these costs (your labor or however much you contribute into the making of the product is also considered into these cost factors). This approach only works, obviously, when your product can be decomposed into components, and each has a cost.

2.2 By supply and demand

If you ever took a course like Economics 101, you may say the simple answer to the pricing of any product is by supply or demand. Another way to say it is let the market decides what the price is. This answer has limitations of course. First, no one knows for sure how the supply and demand curves look like. So to know the price precisely, you would need to model these curves, a non easy task. Second, letting the market decides the price works for a product already on the market. If you have a completely new product (say you just came up with a new way to package a financial derivative), looking into the market for a price is not helpful.

2.3 By LLN

This approach works for a random game, as mentioned in the previous lecture. The basis for it is that we expect *a large number of* people to play the game, thus our revenue from selling the tickets to the game will balance out the profits and losses we make from each individual game. Note that there have to be *two* factors to make this approach works: a lot of incidents of the product (the game) and they are *independent, identically* distributed. Without either, pricing by LLN will not work.

Note: Some people refer to this approach as pricing by expectation. I prefer to call it pricing by LLN, to avoid confusion, since later on we'll see *another* approach of pricing by expectation, which is completely different from this one and NOT based on LLN.

2.4 What about financial products?

Clearly, none of the approach above works for the financial products we mentioned in Section 1. Actually, the reason why the approach by LLN does NOT work is a bit subtle. Even if we have a probabilistic model for the underlying, LLN is still not the right way to price. A more detailed explanation will be given when we come to the one period model.

Here's an important observation to help with the first step: in any financial product, there is a *transfer of risk*, from the buyer of the contract, to the seller of the contract. By selling you a financial product, the contract-seler assume for you the risk of facing the ups an downs of the market price of the underlying. Therefore, the fair price, whatever it is, of the financial product, must properly reflect this risk that the contract seller takes on himself. Hence, risk pricing and / or risk management is often mentioned when one discuss aspects of financial mathematics.

3 Pricing by risk hedging

3.1 Hedging portfolio

Put yourself into the position of the seller of the forward contract in Example 1.1. How much would you charge for such a contract? A better question is: what would you do with the money you obtained from selling the contract, keeping in mind that you have the obligation to sell 1 barrel of oil at 105 USD when November comes? A sensible answer is that you should *invest* that money, so that your position is covered (i.e. you can fulfill the contract at no additional cost to yourself) at the expiration time. But what should you invest in? After some thoughts, you'll see that you should invest some money into the money market (i.e. a saving account with interest) and some money into the oil itself. We call this your *portfolio* (in the money market and in the underlying, which is oil in this case). If your investment strategy is right, at the expiration time, the value of your portfolio would be equal to the payout you have to make, **no matther what the price of the underlying is at that time**. Thus you have completely hedged your risk. We call the portfolio in this case a *hedging portfolio* or a *replicating portfolio* (from the fact that your portfolio "replicates" the value of the financial product at the expiration date).

Since in this scenario, it is clear that you will only have one inflow of money (at the initial time when you sell the contract), we will assume that your portfolio is

self-financing. That is you can re-adjust your position in between the sale time and the expiration time, but you cannot do this using additional funding from outside, nor can you withdraw money from the portfolio.

And so, it is clear that the *fair price* to charge for the contract, is how much it takes to set up your portfolio at the beginning, i.e. the value of your portfolio at the intial time.

Note: the existence of a hedging portfolio is *not guaranteed*. In that case we'll have to find a different way to define a fair price. But if there is a hedging portfolio, then its initial value MUST be the price you charge for the contract. Otherwise, there will be an **arbitrage opportunity**, which we will discuss next.

3.2   Arbitrage opportunity

An arbitrage opportunity is basically a chance to make money without risk. This is clearly an undesirable situation. One of the central principles of math finance is that *arbitrage opportunity cannot exist.* We will call this *the no-arbitrage principle.* If the price of a financial product is such that it allows for an arbitrage opportunity, then it cannot be the right price. On the other hand, if the price of a financial product is such that no arbitrage opportunity can exist, then at least it seems to be the right price to charge ("seem" because the non-existence of no-arbitrage opportunity does not necessarily imply that there exists a hedging portfolio. But we'll see that in most reasonable cases, there is a hedging portfolio whose value equals to the no-arbitrage price).

We will use the hedging portfolio to price a forward contract. But first let's explain why if there exists a hedging portfolio then its initial value must be the price you charge for the contract. Suppose the price for the contract, x, is higher than the value of the hedging portfolio, y. Then you would sell the contract for x, and use y to finance your portfolio. Thus you gain $x - y > 0$ dollars at the beginning, which you can put into a saving account. At the expiration time, because the portfolio is replicating, your position is completely cover. Thus you have made a riskless profit: this is an arbitrage opportunity. This kind of argument is very common in math finance. You should try it yourself, in the case when $x < y$.

3.3 Price of forward contract

We will leave the example of the oil company and work with an abstract setting. Suppose you have a forward contract with expiration T with strike K on an underlying S. That is at time T, you can obtain one share of S for K dollars. Suppose also that the interest rate is r. What is the price for this contract at time $t = 0$? (Note: in this course, *we will always consider the time $t = 0$ as the present time*).

Notation: We will also denote the price of an asset S at time t as S_t. Thus the value of the above forward contract, to the contract holder, at time T is $S_T - K$.

Lemma 3.1. *The price for a forward contract with strike K and expiration T on an underlying S at time $t = 0$ is $S_0 - Ke^{-rT}$.*

Proof.   At time 0, we use the money $S_0 - Ke^{-rT}$ to purchase 1 share of S and borrow $-Ke^{-rT}$ from the bank. Then our initial position is completely balanced. At time T, the value of our portfolio is $S_T - Ke^{-rT}e^{rT} = S_T - K$, which is exactly the value of the forward contract at time T. Thus we have a hedging portfolio whose initial value is $S_0 - Ke^{-rT}$. By what we discussed above, this is the price for the forward contract.

Note: How did we come up with the price $S_0 - Ke^{-rT}$? An explaination will be given below. What you should pay attention to is how the proof is constructed. A typical situation is that from a certain method, we have a candidate for the right price for a contract, and we want to prove it is the right price by constructing a hedging portfolio. The above proof is the first example of this course to show you how to do so.

3.4 Finding the hedging portfolio for a forward contract

The price $S_0 - Ke^{-rT}$ gives us an idea of how to construct a hedging portfolio. But if we do not know this price, how can we proceed? We'll just assume that our portfolio has x shares of S and y dollars in the money market at the beginning. If we can construct x and y so that at the expiration we have

$$xS_T + ye^{rT} = S_T - K,$$

then the price would be $xS_0 + y$.

But note that we have two unknowns x, y but only 1 equation. How can we solve? The key is to note that this equation has to hold, *no matter what S_T is*. It is good

to keep in mind that in this equation, S_T is a **random variable**, while the rest: x, y, K, e^{rT} are all constants. Thus we rewrite the equation as

$$(x - 1)S_T = -ye^{rT} - K.$$

The LHS is a RV, the RHS is a constant. They can only equal if the LHS is also a constant, which means $x = 1$. And easily $y = -Ke^{-rT}$.

Remark: Note that the price of a forward contract we found is **model independent.** That is, we did not make any assumption about the distribution of S_T, or the behavior of S_t from time 0 to T to find this price. This is remarkable, but also because the forward contract is rather simple. To find the price of the Euro options, we will need to build models for S_T to start discussing the price, which is the topic of the next lecture note.

3.5   Forward price

For the forward contract with strike K, there is a special value of K that makes *the value of the contract at the initial time equal to 0*, that is, the buyer does not have to pay money to enter the contract. Note: this does not mean that the contract has no value at the expiration time. In fact its value is still $S_T - K$, and because S_T is random, it cannot be the case that $S_T - K = 0$ with probability 1.

From our result above, we can easily see that this value of K is $S_0 e^{rT}$. This is called the **forward price** of the underlying S (for expiration time T). Again, the forward price (of an asset with expiration T) is the strike price of a forward contract on the same asset with expiration T such that the contract costs no money to enter. You should distinguish this notion from the notion of a forward contract, even though they are clearly related.