

Math 477
Fall 2014
Midterm 2
11/19/2014

Name (Print): _____

This exam contains 4 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit.**
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	15	
3	15	
4	15	
5	15	
6	15	
7	10	
Total:	100	

1. (15 points) The group of friends: Tom, Tim, Tony, Todd, Ted and Jane, June, Jill, Judy and Joan are glad to be featured again in the second Math 477 midterm. To celebrate this distinction, they again go to a restaurant. The two groups arrive at the restaurant independently and uniformly from 7:00 pm to 8:30 pm. What is the probability that one group has to wait for the other group for more than 30 minutes?

Ans: Let X, Y be independent Uniform(1, 90). From a similar picture to the one we drew in class),

$$P(|X - Y| \geq 30) = 2 \frac{(90 - 30)^2}{2} \frac{1}{90^2} = \frac{4}{9}.$$

2. (15 points) The group of the ladies arrive late at the restaurant, which is at the top of the Empire State Building. Being late, they all scramble into the elevator, which has a weight limit of 750 lbs ! We can approximate the day to day weight of the ladies as followed: Jane's as a Normal(150, 5), June's as a Normal(160, 8), Jill's as a Normal(135,6), Judy's as a Normal(170, 10) and Joan's as a Normal(140, 7). (There is a probability that the weights being negative, but it is very close to 0 that it needs not bother you. This is an approximation). What is the probability then, that the elevator is overloaded?

Ans: Let X_1 has $N(150, 5)$, X_2 has $N(160, 8)$, X_3 has $N(135, 6)$ X_4 has $N(170, 10)$ and X_5 has $N(140, 7)$ distribution. Since they are independent, $S = \sum_i X_i$ has $N(755, 36)$ distribution. Then

$$P(S > 750) = P(Z > -\frac{5}{6}) = 1 - P(Z < -\frac{5}{6}) \approx 1 - .2 = .8$$

3. (15 points) The ladies make it safely to the top of the Empire State building. But while they are having dinner, an earthquake hits New York City ! This is a very rare event. Suppose that earthquake hits the New York city area with a rate of once every 5 years. What is the probability that the next earthquake happens more than 3 years from now?

Ans: Let X be the number of earthquake happening in the next 3 years. Then X is distributed as Poisson(3/5). Thus

$$P(X = 0) = e^{-3/5}.$$

4. Surviving the earthquake unharmed, the group of friends celebrate again by going to the store and buy Iphone 6. Suppose that the life time (in years) of an Iphone 6 is distributed as an Exponential(1/5) random variable and the distribution of the individual life times are independent.

- (a) (5 points) Let S represent the total life time of the Iphones these friends have (there are 10 Iphones total). What is the probability that S is at least 50? (You only have to write down an integral here, without computing the actual integral).

Ans: S has a Gamma(10, 1/5) distribution. Therefore

$$P(S \geq 50) = \int_{50}^{\infty} \frac{1/5 x^9 e^{-1/5x}}{9!}.$$

- (b) (10 points) What is the probability that at most 3 iPhones in this group will malfunction before 3 years? (An exact expression is required here, no approximation or actual numerical value needed).

Ans: The probability that any particular iPhone malfunction before 3 years is

$$p = \int_0^3 \frac{1}{5} e^{-1/5x} dx = 1 - e^{-3/5}.$$

Therefore the number of iPhone malfunctioning before 3 years, X , has a Bin(10,p) distribution. Thus

$$P(X \leq 3) = \sum_{i=0}^3 \binom{10}{i} p^i (1-p)^{10-i}.$$

5. (15 points) Wanting to pursue higher education, Tim, Tony and Todd decide to apply to graduate school at Harvard, MIT and Princeton. Suppose that each of them independently choose only ONE school among these 3 to apply to. Let us call a school that none of these gentlemen applies to a terrible school. What is the expected number of the terrible schools?

Ans: Let X be the number of terrible schools. Then X has the following distribution:

$$\begin{aligned} P(X=0) &= \left(\frac{1}{3}\right)^3 3! \\ P(X=1) &= \left(\frac{1}{3}\right)^3 \binom{3}{2} \binom{3}{2} 2 \\ P(X=2) &= \left(\frac{1}{3}\right)^3 \binom{3}{1}. \end{aligned}$$

Explanation of the term $P(X=1)$: We need to choose which 2 schools they all apply to, then we need to choose which 2 guys to go to one of those school. The order of which 2 go to which school matter (Tim Tony goes to Harvard, Todd goes to Princeton is different from Todd goes to Harvard and Tim Tony goes to Princeton). Thus we need to multiply the number of choices by 2 as well.

Now the expectation is just

$$E(X) = \frac{18}{27} + 2 \frac{3}{27} = \frac{24}{27}.$$

6. The graduate admission officer looks at the past midterm scores of Tony (X) and Todd (Y), which can be approximated by a joint continuous distribution with the joint density as followed:

$$\begin{aligned} f_{XY}(x,y) &= cxy, \quad 75 \leq x, y \leq 100, \quad 160 \leq x+y \leq 200 \\ &= 0, \quad \text{otherwise.} \end{aligned}$$

- (a) (5 points) Find c so that f_{XY} is a joint density.

Ans: We require

$$\int_{75}^{85} \int_{160-y}^{100} xy dx dy + \int_{85}^{100} \int_{75}^{100} xy dx dy = \frac{1}{c}.$$

The first integral evaluates to

$$\begin{aligned} \frac{1}{2} \int_{75}^{85} [100^2 - (160 - y)^2] y dy &= 5000(85 - 75) - \int_{75}^{85} (160 - y)^2 y dy \\ &= 5 \times 10^4 - \int_{75}^{85} (160^2 - 320y + y^2) y dy \\ &= 5 \times 10^4 - \left(\frac{160^2}{2} (85^2 - 75^2) - \frac{320}{3} (85^3 - 75^3) + \frac{85^4 - 75^4}{4} \right). \end{aligned}$$

The second integral evaluates to

$$\int_{85}^{100} \int_{75}^{100} xy dx dy = \frac{1}{4} (100^2 - 85^2) (100^2 - 75^2).$$

- (b) (10 points) The officer suspects that Tony and Todd did not work independently on their midterms. Is he paranoid?

Ans: No he's not paranoid. The form of f_{XY} does NOT factor into $f_X(x)$ and $f_Y(y)$ since the region involves $x + y \geq 160$.

7. (10 points) The graduate officer decides to give Tony and Todd a second chance by posing the following problem. His wife is currently driving a BMW, whose life time (in years) X is distributed as Exponential(1/10). The officer, on the other hand, is driving an old Volkswagen inherited from his grandparents, whose remaining life time Y is distributed as Exponential(1). Suppose X, Y are independent. He wants to find out how much his wife's car will outlast his own by looking at the random variable $Z = \frac{X}{Y}$. What then is the density of Z ?

Ans: For $z \geq 0$

$$\begin{aligned} P(Z \leq z) &= P\left(\frac{X}{Y} \leq z\right) = P(X \leq zY) = 1/10 \int_0^\infty \int_0^{zy} e^{-1/10x} e^{-y} dx dy \\ &= \int_0^\infty e^{-y} - e^{-(z/10+1)y} dy \\ &= 1 - \frac{1}{\frac{z}{10} + 1}. \end{aligned}$$

Thus

$$f_Z(z) = \frac{1/10}{\left(\frac{z}{10} + 1\right)^2}.$$