Math 477
Fall 2014
Midterm 1
10/13/2014

Name (Print): ______________________

This exam contains 6 pages (including this cover page) and 7 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	15	
2	15	
3	15	
4	10	
5	15	
6	15	
7	15	
Total:	100	

1. (15 points) Five Math 477 buddies Tom, Tim, Tony, Todd and Ted go to a restaurant to celebrate finishing their first midterm. They can choose among 7 different entreé options: steak, pork ribs, lobster, salmon, chicken, lamb and vegetarian. Suppose each choose 1 entreé randomly among the 7 options and the choices are equally likely. What is the probability that there are exactly 3 different entreé types in their dinner?

   Ans: The sample size is 7^5. The number of ways they can choose different entreé is

$$\binom{7}{3}\left(3^5 - [\binom{3}{2}(2^5-2)] - 3\right).$$

   Thus the probability is

$$\frac{\binom{7}{3}\left(3^5 - [\binom{3}{2}(2^5-2)] - 3\right)}{7^5}.$$

2. (15 points) After dinner, Tom, Tim, Tony, Todd and Ted go to ball room dancing with Jane, June, Jill, Judy and Joan, who are also in Math 477. Each guy has one (and only one !) girl they have romantic interest on: Tom likes June, Tim likes Jill, Tony likes Judy, Todd likes Jane and Ted likes Joan. However, for their first dance, the host matches a guy randomly with a girl. What is the probability that only one guy gets to dance with the girl he likes in the first dance?

   Ans: Let E_i be the event that the ith guy gets to dance with the girl he likes, $i = 1 \cdots 5$ and F be the event that only one guy gets to dance with the girl he likes. Then

$$P(F) = \sum_{i=1}^{5} P(F|E_i)P(E_i).$$

   Now $P(E_1) = P(E_2) = \cdots = P(E_5) = 1/5$. And by symmetry

   $P(F|E_1) = P(F|E_2) = \cdots = P(F|E_5) = P($ no 4 guy gets to dance with the girl he likes $)$.

   We compute the probability that at least 1 guy gets to dance with the girl he likes. Let A_i be the event that the ith guy gets to dance with the girl he likes, $i = 1, \cdots, 4$.

$$\begin{aligned} P(\cup_{i=1}^4 A_i) &= \sum_{i=1}^{4} P(A_i) - \sum_{i<j} P(A_iA_j) + \sum_{i<j<k} P(A_iA_jA_k) - P(A_1A_2A_3A_4) \\ &= 1 - 6\frac{2!}{4!} + 4\frac{1}{4!} - \frac{1}{4!}. \end{aligned}$$

   Thus P(no 4 guy gets to dance with the girl he likes) is

$$6\frac{2!}{4!} - 4\frac{1}{4!} + \frac{1}{4!}.$$

Thus

$$\begin{aligned} P(F) &= \frac{1}{5}5\left(6\frac{2!}{4!} - 4\frac{1}{4!} + \frac{1}{4!}\right) \\ &= 6\frac{2!}{4!} - 4\frac{1}{4!} + \frac{1}{4!}. \end{aligned}$$

3. (15 points) After dancing, the group of 10 friends have to help the host clean up. Out of 10 people available, the host randomly selects 4 people to help her clean up (her choices are equally likely among the 10 available).

a) What is the probability that among the 4 people she selects, there is no couple with romantic interest? (Note that the host can select anything from 4 boys to 3 boys and 1 girl to 4 girls).

Ans: The sample size is $\binom{10}{4}$.

The only way she can select 4 people without romantic interest is to select randomly 4 couples, and among each couple randomly choose 1 person. That is:

$$\binom{5}{4}2^4.$$

So the probability is

$$\frac{\binom{5}{4}2^4}{\binom{10}{4}}.$$

b) What is the probability that she selects 2 boys and 2 girls, and among them there is no couple with romantic interest?

The only way she can select 2 boys and 2 girls, among them there is no couple with romantic interest is first to select 2 couples randomly among the 5, and choose either 2 boys or 2 girls from these 2 couples, then randomly select 2 couples from the remaining 3, and choose 2 girls (if at first she chose 2 boys) or 2 boys (if at first she chose 2 girls) from these 2 couples. That is:

$$2\binom{5}{2}\binom{3}{2}.$$

Thus the probability is

$$\frac{2\binom{5}{2}\binom{3}{2}}{\binom{10}{4}}.$$

4. (10 points) After the dance, Tom goes home. Being an un-organized person, there is 30% probability that Tom puts his house key in his car, 50% probability that he puts his key in his jacket and 20% probability that he lost his key. If the key is in the car, there is 40% chance it's in the glovebox and 60% chance it's in the car door. If the key is in his jacket, there is 30% chance it's in his right pockect and 70% chance it's in his left pocket. Upon arriving home, Tom checks his glovebox and found no key. What is the conditional probability, based on this event, that he lost his key?

Let E be the event key is in car, F key in jacket, G key is lost. E_1 key in glovebox, E_2 key in car door. F_1 key in left pocket, F_2 key in right pocket.

Then

$$\begin{aligned} P(G|E_1^c) &= \frac{P(GE_1^c)}{P(E_1^c)} \\ &= \frac{P(G)}{P(E_1^c)}, \end{aligned}$$

where $P(GE_1^c) = P(G)$ because if he lost his key it's not in the glovebox. And

$$P(E_1^c) = P(E_1^c|E)P(E) + P(E_1^c|E^c)P(E^c) = (.6)(.3) + .7 = .88$$

Thus $P(G|E_1^c) = \frac{.2}{.88}$.

5. (15 points) Tim, on the other hand, always likes to plan ahead. Because he is having a great experience in his 477 course, Tim wants to be an actuary. He plans to take the first 3 actuarial exam in the coming summer. He will take the first one in June. If he passes the 1st, he will take the 2nd in July. If he passes the 2nd, he will take the 3rd in August. If he fails an exam, he is not allowed to take any others. The probability that he passes the 1st exam is .8. If he passes the 1st, then the conditional probability that he passes the 2nd is .9. If he passes both the 1st and 2nd, then the conditional probability that he passes the 3rd is .7.

a. What is the probability that he passes all 3 exams?

Let E_i be the event that he passes exam $i, i = 1, 2, 3$. Then

$$P(E_1E_2E_3) = P(E_3|E_1E_2)P(E_2|E_1)P(E_1) = (.7)(.9)(.8).$$

b. Given that he did not pass all three exams, what is the conditional probability that he failed the 2nd exam?

$$P(E_1E_2^c|(E_1E_2E_3)^c) = \frac{P(E_1E_2^c, (E_1E_2E_3)^c)}{P((E_1E_2E_3)^c)}.$$

If he did not pass the second, he would not pass all 3, that is

$$P(E_1E_2^c, (E_1E_2E_3)^c) = P(E_1E_2^c) = P(E_2^c|E_1)P(E_1) = (.1)(.8).$$

Thus

$$P(E_1E_2^c|(E_1E_2E_3)^c) = \frac{(.1)(.8)}{1 - (.7)(.9)(.8)}.$$

6. (15 points) Jill, hearing about Tim's great knowledge in probability, consults with him about an insurance question. It is known that during any given year, an accident prone person will have an accident with probability .4, whereas the corresponding figure for a person who is not accident prone is .2. It is also known that 30% of the population is accident prone.

a. What is the probability that a Jill will have an accident during any given year?

Let E be the event that Jill is accident prone and F the event that she has an accident. Then

$$\begin{aligned} P(F) &= P(F|E)P(E) + P(F|E^c)P(E^c) \\ &= (.4)(.3) + (.2)(.7). \end{aligned}$$

b. It turns out that Jill just had an accident last year. Suppose that the probability that an accident prone person will have an accident is always .4, regardless of their accident history. Also the probability that an non-accident prone person will have an accident is always .2, regardless of their accident history. What then, is the conditional probability that Jill will have an accident this year, given that she had an accident last year?

Let F_1 be the event Jill has an accident last year, F_2 the event she will have an accident this year. Then

$$\begin{aligned} P(F_2|F_1) &= P(F_2|F_1E)P(E|F_1) + P(F_2|F_1E^c)P(E^c|F_1) \\ &= P(F_2|E)P(E|F_1) + P(F_2|E^c)P(E^c|F_1) \end{aligned}$$

because accident likelihood does not depend on accident history, given a person's tendency as in the hypothesis.

Now

$$P(E|F_1) = P(EF_1)/P(F_1) = P(F_1|E)P(E)/P(F_1) = \frac{(.4)(.3)}{(.4)(.3) + (.2)(.7)}.$$

Thus

$$P(E^c|F_1) = 1 - \frac{(.4)(.3)}{(.4)(.3) + (.2)(.7)}.$$

We also have $P(F_2|E) = .4$ and $P(F_2|E) = .2$). Plug in these numbers give the answer.

7. (15 points) The group of 10 friends choose 2 people randomly among themselves to attend a national conference in probability (the choices are equally likely). To encourage gender diversity, the conference will award the attendees 100 dollars if the 2 attendees are of different gender; and they would charge 110 dollars if the 2 attendees are of the same gender. Let X be the random variable that represents the amount that the group receives from the conference (that is $X = -110$ if the group representatives are of the same gender.) Find $E(X)$ and $Var(X)$.

Let E be the event that two attendees are of different genders. Then

$$P(E) = \frac{5^2}{\binom{10}{2}}.$$

Thus

$$E(X) = 100\frac{5^2}{\binom{10}{2}} - 110(1 - \frac{5^2}{\binom{10}{2}}).$$

We also have

$$E(X^2) = 100^2\frac{5^2}{\binom{10}{2}} + 110^2(1 - \frac{5^2}{\binom{10}{2}}).$$

Then

$$Var(X) = E(X^2) - E^2(X).$$