Moment generating functions

Math 477

November 26, 2014

1 Introduction

What information do we need to completely characterize a RV? For a discrete RV we need its ditribution mass function, for a continuous RV we need its density. These are not the only ways. An alternative piece of information we can use is the Laplace transform of the distribution mass function or the density fuction. In the probabilistic context, we call it the moment generating function. Because the Laplace transform has an inverse, knowing the moment generating function is the same as knowing the RV. As you will see, in certain situations, it is more convenient to have information about the moment generating function, rather than the density function itself.

2 Definition

The moment generating function of a RV X is defined as

$$\begin{aligned} M(t) &= E(e^{tX}) \\ &= \sum_i e^{ti} P(X = i) \text{ if X is discrete} \\ &= \int_{-\infty}^{\infty} e^{tx} f_X(x) dx \text{ if X is continuous.} \end{aligned}$$

Note that $M(t)$ may NOT be defined for all values of t (the value may be ∞). A simple example is for the exponential RV (λ) with $t > \lambda$:

$$\begin{aligned} E(e^{tX}) &= \int_0^{\infty} e^{tx} \lambda e^{-\lambda x} \\ &= \int_0^{\infty} \lambda e^{(t-\lambda)x} dx = \infty. \end{aligned}$$

3 Properties

3.1 Moment generating

Under certain condition, we can obtain the moments of the RV from the moment generating function, as followed:

$$E(X^n) = M^{(n)}(0).$$

The reasoning is intuitively as followed:

$$\frac{d}{dt}M(t) = \frac{d}{dt}E(e^{tX}) = E(\frac{d}{dt}e^{tX}) = E(Xe^{tX}).$$

Thus

$$\frac{d}{dt}M(t)|_{t=0} = E(X)/$$

The condition that would make this computation valid is so that we can switch the order of summation (or integration) with differentiation with respect to t.

3.2 Unique determination of distribution

Let X and Y be 2 RVs with moment generating functions $M_X(t)$ and $M_Y(t)$. (Technically we only require them to exist and be finite for some region around $t = 0$). If $M_X(t) = M_Y(t)$ then X and Y have the same distribution.

For example, later on we will compute the moment generating function of a $N(\mu, \sigma^2)$ to be $M(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$. So if we have a RV Y such that its moment generating function is $M_Y(t) = e^{\frac{t^2}{2}}$ then we know Y must have standard Normal distribution.

3.3 Moment generating function of sum of independent RVs

Let X, Y be independent. Then

$$M_{X+Y}(t) = E(e^{t(X+Y)}) = E(e^{tX}e^{tY}) = E(e^{tX})E(e^{tY}) = M_X(t)M_Y(t).$$

This gives us a very elegant way to compute the moment generating function of the sum, knowing the moment generating function of the individual summand. For example, again assuming we know $M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}$ for X having Normal(μ, σ^2) distribution. Let $X_i, i = 1, 2$ have Normal(μ_i, σ_i^2). Then

$$M_{X_1+X_2} = e^{\mu_1 t + \frac{\sigma_1^2 t^2}{2}} e^{\mu_2 t + \frac{\sigma_2^2 t^2}{2}} = e^{(\mu_1+\mu_2)t + \frac{(\sigma_1^2+\sigma_2^2)t^2}{2}}.$$

That is, $X_1 + X_2$ have Normal $(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$ distribution, as we learned before.

4 Moment generating functions of specific RVs

Here we just list the moment generating functions of various RVs we dealt with so far in this course. The derivation of these moment generating functions can be found in the textbook.

4.1 Discrete

4.1.1 Binomial(n,p)

$$M_X(t) = (pe^t + 1 - p)^n.$$

Remark: From the formula, it is clear that the sum of 2 independent Bin(n,p) and Bin(m,p) has distribution Bin(n+m,p).

4.1.2 Possion(λ)

$$M_X(t) = e^{\lambda(e^t - 1)}.$$

Remark: From the formula, it is clear that the sum of 2 independent Poisson λ_1 and λ_2 has distribution Poisson($\lambda_1 + \lambda_2$).

4.1.3 Geometric p

$$M_X(t) = \frac{pe^t}{1 - (1-p)e^t}.$$

Remark: From the formula, it is clear that the sum of 2 independent Geometric p is NOT a geometric p since the product would be

$$M_{X+Y}(t) = \left(\frac{pe^t}{1 - (1-p)e^t}\right)^2.$$

It follows from the next result that it is a negative Binomial $2, p$.

4.1.4 Negative Binomial r, p

$$M_X(t) = \left(\frac{pe^t}{1 - (1-p)e^t}\right)^r.$$

Remark: From the formula, it is clear that the sum of 2 independent Negative Binomial r_1, p and r_2, p is a Negative Binomial $r_1 + r_2, p$.

4.2 Continuous

4.2.1 Uniform $[a, b]$

$$M_X(t) = \frac{e^{tb} - e^{ta}}{t(b-a)}.$$

Remark: From the formula, it is clear that the sum of 2 independent X, Y having Uniform $[a, b]$ distribution doest NOT have Uniform distribution since

$$M_{X+Y}(t) = \left(\frac{e^{tb} - e^{ta}}{t(b-a)}\right)^2.$$

4.2.2 Exponential λ

$$M_X(t) = \frac{\lambda}{\lambda - t}.$$

Remark: From the formula, it is clear that the sum of 2 independent X, Y having Exponential(λ) distribution is NOT an Exponential since

$$M_{X+Y}(t) = \left(\frac{\lambda}{\lambda - t}\right)^2.$$

It is a Gamma distribution as the next result will show.

4.2.3 Gamma α, λ

$$M_X(t) = \left(\frac{\lambda}{\lambda - t}\right)^\alpha.$$

Remark: From the formula, it is clear that the sum of 2 independent X, Y having Gamma(α_1, λ) and Gamma(α_2, λ) distribution has a Gamma ($\alpha_1 + \alpha_2, \lambda$) distribution.

4.2.4 Normal μ, σ^2

$$M_X(t) = e^{\mu t + \frac{\sigma^2 t^2}{2}}.$$

Remark: From the formula, it is clear that the sum of 2 independent X, Y having Normal(μ_1, σ_1^2) and Normal(μ_2, σ_2^2) distribution is a Normal ($\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2$).