Joint distribution of Functions of RVs

Math 477

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1 Introduction

Let X_1, X_2 have joint density $f_{X_1X_2}(x_1, x_2)$. Let $g_1, g_2 : \mathbb{R}^2 \to \mathbb{R}$ be 2 functions. Denote $Y_i = g_i(X_1, X_2)$. Intuitively, Y_1, Y_2 would be jointly continuous with some density function $f_{Y_1Y_2}(y_1, y_2)$ We want to find what this density $f_{Y_1Y_2}$ is. Our experience with functions of a single RV before tells us that there should be some condition on g_1, g_2, more specifically something along the line of invertibility. Otherwise, given y_1, y_2 it is complicated to solve for x_1, x_2 such that $y_i = g_i(x_1, x_2)$ and it's hard to give a general formula for $f_{Y_1Y_2}$ in such situation. More specifically, assume g_i satisfies the following:

1. The function g_i is jointly invertible: For any pair y_1, y_2, there exists a unique pair x_1, x_2 such that $y_i = g_i(x_1, x_2)$. Note: This does NOT require that each g_i is INDIVIDUALLY invertible. For example, let

$$\begin{aligned} g_1(x_1, x_2) &= x_1 + x_2 \\ g_2(x_1, x_2) &= x_1 - x_2, \end{aligned}$$

then they are jointly, but not individually invertible.

2. The determinant of the Jacobian matrix of the mapping $g(x_1, x_2) = \begin{bmatrix} g_1(x_1, x_2) \\ g_2(x_1, x_2) \end{bmatrix}$ is non-zero for all x_1, x_2. Simply put, we require:

$$J(x_1, x_2) := \frac{\partial g_1}{x_1}\frac{\partial g_2}{x_2} - \frac{\partial g_1}{x_2}\frac{\partial g_2}{x_1} \neq 0.$$

Then we have the following result

Theorem 1.1. *X_1, X_2 have joint density $f_{X_1X_2}(x_1, x_2)$. Let g_1, g_2 satisfy the conditions 1,2 above. Let $Y_i = g_i(X_1, X_2), i = 1, 2$. Denote $g^{-1}(y_1, y_2)$ as the pair of x_1, x_2*

such that $y_i = g_i(x_1, x_2)$. *Then* Y_1, Y_2 *are jointly continuous and their joint density is given as*

$$f_{Y_1Y_2}(y_1, y_2) = \frac{f_{X_1X_2}(g^{-1}(y_1, y_2))}{\left|J\big(g^{-1}(y_1, y_2)\big)\right|}.$$

2 Examples and applications

Example 2.1. *Sum and difference of RVs*

If we let

$$\begin{aligned} g_1(x_1, x_2) &= x_1 + x_2 \\ g_2(x_1, x_2) &= x_1 - x_2, \end{aligned}$$

then it is clear that $J(x_1, x_2) = -2$ *for all* x_1, x_2. *Also*

$$g^{-1}(y_1, y_2) = (\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}).$$

Therefore,

$$f_{Y_1Y_2} = \frac{1}{2} f_{X_1X_2}(\frac{y_1 + y_2}{2}, \frac{y_1 - y_2}{2}).$$

Example 2.2. *Sum and difference of independent Uniform*

Let X_1, X_2 *have joint independent Uniform[0,1] distribution. Then from the example above, we have*

$$\begin{aligned} f_{Y_1Y_2} &= \frac{1}{2} \text{ if } 0 \leq \frac{y_1 + y_2}{2} \leq 1 \text{ and } 0 \leq \frac{y_1 - y_2}{2} \leq 1; \\ &= 0 \text{ otherwise.} \end{aligned}$$

This region is a square with vertices at $(0, 0), (1, 1), (2, 0), (1, -1)$. *Thus we can compute the marginal density of* y_1 *easily as:*

$$\begin{aligned} f_{Y_1}(y_1) &= \frac{1}{2} 2y_1 = y_1, 0 \leq y_1 \leq 1 \\ f_{Y_1}(y_1) &= \frac{1}{2} 2(2 - y_1) = 2 - y_1, 1 \leq y_1 \leq 2. \end{aligned}$$

From a similar technique one can also derive the marginal distribution of Y_2, that is the difference of 2 independent Uniforms.

Example 2.3. *Sum and difference of independent Normal*

Let X_1, X_2 have joint independent standard Normal distribution. Then from the example above, we have

$$\begin{aligned} f_{Y_1Y_2} &= \frac{1}{4\pi}e^{-\frac{(y_1+y_2)^2}{8}-\frac{(y_1-y_2)^2}{8}} \\ &= \frac{1}{4\pi}e^{-\frac{y_1^2}{4}}e^{-\frac{y_2^2}{4}}. \end{aligned}$$

Thus we see that not only INDIVIDUALLY, the sum and difference of standard Normals are Normally distributed with mean 0 and Variance 2, but they are jointly INDEPDENDENT as well.

Example 2.4. *Sum and difference of independent Exponential*

Let X_1, X_2 have joint independent Exponential distribution with rate λ_1, λ_2, respectively. Then from the example above, we have

$$\begin{aligned} f_{Y_1Y_2} &= \frac{\lambda_1\lambda_2}{2}e^{-\lambda_1(\frac{y_1+y_2}{2})-\lambda_2(\frac{y_1-y_2}{2})} \text{ if } y_1+y_2 \geq 0, y_1-y_2 \geq 0 \\ &= 0 \text{ otherwise.} \end{aligned}$$