Sum of independent continuous RVs

Math 477

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1 Sums of continuous RVs

1.1 Introduction

We want to obtain a density for the random variable $Z = X + Y$, where X, Y are independent continuous RVs with densities f_X, f_Y respectively.

1.2 The general formula

We have

$$
\begin{aligned}
F_{X+Y}(z) &= P(X + Y \leq z) \\
&= \iint_{x+y \leq z} f_{XY}(x,y)dxdy \\
&= \int_{-\infty}^{\infty} \int_{-\infty}^{z-y} f_X(x) f_Y(y) dxdy \\
&= \int_{-\infty}^{\infty} f_Y(y) \int_{-\infty}^{z-y} f_X(x) dxdy \\
&= \int_{-\infty}^{\infty} F_X(z-y) f_Y(y) dy.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
f_Z(z) &= \frac{d}{dz} \int_{-\infty}^{\infty} F_X(z-y) f_Y(y) dy \\
&= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy. \qquad (1)
\end{aligned}
$$

We say $f_Z(z)$ is the convolution of f_X, f_Y. Note that we also have

$$
f_Z(z) = \int_{-\infty}^{\infty} f_X(x) f_Y(z-x) dx.
$$

1.3 Examples

1.3.1 Sum of Uniforms

Example 1.1. *Let X and Y be independent Uniform[0,1] RVs. Find the density of $Z = X + Y$.*

Ans: Before we even attempt the solution, you should note that Z is takes values in the range $[0, 2]$. Applying the above formula gives

$$\begin{aligned} f_Z(z) &= \int_{-\infty}^{\infty} f_X(z-y) f_Y(y) dy \\ &= \int_0^1 \mathbf{1}_{0 \le z-y \le 1} f_Y(y) dy. \end{aligned}$$

We utilize the fact that $0 \le z \le 2$ here. The inequality $0 \le z - y \le 1$ implies that

$$z - 1 \le y \le z.$$

So we need to compare $z-1$ with 0 and z with 1, as those are the lower and upper limits of our original integral.

If $0 \le z \le 1$ then

$$f_Z(z) = \int_0^z 1 dy = z.$$

If $1 \le z \le 2$ then

$$f_Z(z) = \int_{z-1}^1 1 dy = 1 - (z-1) = 2 - z.$$

In summary, the sum of 2 indendent Uniforms [0,1] has density

$$\begin{aligned} f_Z(z) &= z; 0 \le z \le 1 \\ &= 2 - z; 1 \le z \le 2. \end{aligned}$$

1.3.2 Sum of Gammas

Proposition 1.2. *Let X, Y be independent Gamma with parameters (α, λ) and (β, λ), respectively. Then $X + Y$ has distribution Gamma $(\alpha + \beta, \lambda)$.*

Proof. Recall that the density of Gamma(α, λ) is

$$f_X(x) = \frac{\lambda e^{-\lambda x} (\lambda x)^{\alpha - 1}}{\Gamma(\alpha)}, x \ge 0.$$

Thus by the formula (1)

$$f_Z(z) = \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_{-\infty}^{\infty} \lambda e^{-\lambda(z-y)}(\lambda(z-y))^{\alpha-1}\lambda e^{-\lambda y}(\lambda y)^{\beta-1}dy.$$

The requirement that $x = z - y \geq 0$ reduces the above integral to

$$f_Z(z) = Ke^{-\lambda z}\int_0^z (z-y)^{\alpha-1}y^{\beta-1}dy,$$

where we absorb all constants related to α, β, λ into K. The reason is only the form of the density is important. The constant has to work out right for the integral of the density to be 1, since we know a priori that $f_Z(z)$ is a density.

Now

$$\begin{aligned}\int_0^z (z-y)^{\alpha-1}y^{\beta-1}dy &= z^{\alpha-1}\int_0^z (1-\frac{y}{z})^{\alpha-1}y^{\beta-1}dy \\ &= z^{\alpha+\beta-1}\int_0^1 (1-u)^{\alpha-1}u^{\beta-1}du,\end{aligned}$$

by the substitution $u = \frac{y}{z}, du = \frac{1}{z}dy$. Since $\int_0^1 (1-u)^{\alpha-1}u^{\beta-1}du$ is just another constant, it can be absorbed into K to give us

$$f_Z(z) = Ke^{-\lambda z}z^{\alpha+\beta-1},$$

which is the right form for a Gamma $(\alpha+\beta, \lambda)$. The proof is finished.

Corollary 1.3. *Let $X_1, X_2, \cdots, X_n$ be independent Exponential (λ). Then $Y = X_1 + X_2 + \cdots + X_n$ has Gamma (n, λ) distribution.*

Proof. Since Exponential(λ) = Gamma $(1, \lambda)$, by the above Proposition, we have $X_1 + X_2$ has distribution Gamma $(2, \lambda)$. But $X_1 + X_2$ are independent of X_3, thus $X_1 + X_2 + X_3$ has distribution Gamma $(3, \lambda)$. Continue in this fashion we have the result.

1.3.3 Sum of Normals

Proposition 1.4. *Let X_i be independent Normal with parameters $(\mu_i, \sigma_i^2), i = 1, 2$ respectively. Then $X + Y$ has distribution Normal $(\mu_1 + \mu_2, \sigma_1^2 + \sigma_2^2)$.*

Proof. Since we have showed that $X_i - \mu_i$ have Normal$(0, \sigma_i^2)$ distribution, WLOG we can assume $\mu_1 = \mu_2 = 0$. By a similar reasoning, we can assume $\sigma_2 = 1$.

Now since only form matters but not constants, we have

$$f_X(z-y)f_Y(y) = Ke^{-\frac{(z-y)^2}{2\sigma^2}}e^{-\frac{y^2}{2}}.$$

Completing the square we have

$$\begin{aligned}(z-y)^2 + \sigma^2 y^2 &= (1+\sigma^2)y^2 - 2zy + z^2 \\ &= \Big(\sqrt{1+\sigma^2}y - \frac{z}{\sqrt{1+\sigma^2}}\Big)^2 + (1 - \frac{1}{1+\sigma^2})z^2 \\ &= (1+\sigma^2)\Big(y - \frac{z}{1+\sigma^2}\Big)^2 + \frac{\sigma^2}{1+\sigma^2}z^2.\end{aligned}$$

Therefore,

$$f_X(z-y)f_Y(y) = Ke^{-\frac{1+\sigma^2}{2\sigma^2}\Big(y-\frac{z}{1+\sigma^2}\Big)^2}e^{-\frac{z^2}{2(1+\sigma^2)}}.$$

Integrating the above in y just leaves us with

$$f_Z(z) = Ke^{-\frac{z^2}{2(1+\sigma^2})}.$$

This is the form of a Normal$(0, 1+\sigma^2)$. The proof is complete.

Corollary 1.5. *Let $X_1, X_2, \cdots, X_n$ be independent Normal (μ_i, σ_i^2). Then $Y = X_1 + X_2 + \cdots + X_n$ has Normal $(\sum_i \mu_i, \sum_i \sigma_i^2)$ distribution.*

1.3.4 The Chi-square distribution

Definition 1.6. *Let $Y = Z^2$, where Z has standard normal distribtion. Then we say Y has the Chi-square distribution with 1 degree of freedom.*

Let $Y_1, Y_2, \cdots, Y_n$ be independent Chi-square distribution with one degree of freedom. Then we say $Y = \sum_{i=1}^n Y_i$ has the Chi-square distribution with n degrees of freedom.

We have:

$$P(Y \le y) = P(-\sqrt{y} \le Z \le \sqrt{y}) = 2P(0 \le Z \le \sqrt{y}).$$

Therefore,

$$f_Y(y) = \frac{1}{\sqrt{y}}f_Z(\sqrt{y}) = \frac{y^{1/2-1}e^{-y/2}}{\sqrt{2\pi}}.$$

That is, a Chi-square RV with one degree of freedom is just a Gamma(1/2,1/2) RV. From our result for summing independent Gamma RVs, a Chi-square RV with n degrees of freedom is just a Gamma(n/2,1/2) RV.

2 Sums of discrete RVs

2.1 The formula

Let X, Y be independent discrete RVs. Let $Z = X + Y$. Then

$$\begin{aligned}
P(Z=k) = P(X+Y=k) &= \sum_{i,j:i+j=k} P(X=i, Y=j) \\
&= \sum_{i,j:i+j=k} P(X=i)P(Y=j) \\
&= \sum_i P(X=i)P(Y=k-i) \\
&= \sum_j P(X=k-j)P(Y=j).
\end{aligned}$$

2.2 Sum of Poissons

Proposition 2.1. *Let X_i be independent Poisson with parameters $\lambda_i, i = 1, 2$ respectively. Then $X + Y$ has distribution Poisson $\lambda_1 + \lambda_2$.*

Proof. Let $Z = X + Y$. Note that Z takes values $0, 1, 2 \cdots$. Then

$$\begin{aligned}
P(Z=k) &= \sum_i P(X=i)P(Y=k-i) \\
&= \sum_i e^{-\lambda_1}\frac{\lambda_1^i}{i!}e^{-\lambda_2}\frac{\lambda_2^{k-i}}{k-i!} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{k!}\sum_{i=0}^k \binom{k}{i}\lambda_1^i\lambda_2^{k-i} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{k!}(\lambda_1+\lambda_2)^k \sum_{i=0}^k \left(\binom{k}{i}\frac{\lambda_1}{\lambda_1+\lambda_2}\right)^i \left(\frac{\lambda_2}{\lambda_1+\lambda_2}\right)^{k-i} \\
&= \frac{e^{-(\lambda_1+\lambda_2)}}{k!}(\lambda_1+\lambda_2)^k.
\end{aligned}$$

Corollary 2.2. *Let $X_1, X_2, \cdots, X_n$ be independent Poisson λ_i. Then $Y = X_1 + X_2 + \cdots + X_n$ has Poisson $\sum_i \lambda_i$ distribution.*

2.3 Sum of Binomials

Proposition 2.3. *Let X_i be independent Binomial with parameters $\nu_i, i = 1, 2$ respectively and success probability p . Then $X + Y$ has distribution Binomial $(\nu_1 + \nu_2, p)$.*

Proof. Let $Z = X + Y$. Note that Z takes values $0, 1, 2 \cdots, n_1 + n_2$. Then

$$\begin{aligned} P(Z = k) &= \sum_i P(X = i)P(Y = k - i) \\ &= \sum_{i=0}^{k} \binom{n_1}{i} p^i (1-p)^{n_1 - i} \binom{n_2}{k-i} p^{k-i} (1-p)^{n_2 - (k-i)} \\ &= p^k (1-p)^{n_1 + n_2 - k} \sum_{i=0}^{k} \binom{n_1}{i} \binom{n_2}{k-i}. \end{aligned}$$

We claim that

$$\sum_{i=0}^{k} \binom{n_1}{i} \binom{n_2}{k-i} = \binom{n_1 + n_2}{k},$$

via the following argument: to pick k objects out of $n_1 + n_2$ total, we just have to pick i objects out of the first n_1, $k - i$ objects out of n_2, where $i = 0, 1, \cdots, k$.

2.4 Sum of Negative Binomial

Proposition 2.4. *Let X_i be independent Negative Binomial with parameters $r_i, i = 1, 2$ respectively and success probability p . Then $X + Y$ has distribution Negative Binomial $(r_1 + r_2, p)$.*

Proof. Let $Z = X + Y$. Note that Z takes values $r_1 + r_2, r_1 + r_2 + 1, r_1 + r_2 + 2 \cdots$. Then for $k \geq r_1 + r_2$

$$\begin{aligned} P(Z = k) &= \sum_i P(X = i)P(Y = k - i) \\ &= \sum_{i=r_1}^{k-r_2} \binom{i-1}{r_1 - 1} p^{r_1} (1-p)^{i - r_1} \binom{k-i-1}{r_2 - 1} p^{r_2} (1-p)^{k-i-r_2} \\ &= p^{r_1 + r_2} (1-p)^{k - (r_1 + r_2)} \sum_{i=r_1}^{k-r_2} \binom{i-1}{r_1 - 1} \binom{k-i-1}{r_2 - 1} \\ &= p^{r_1 + r_2} (1-p)^{k - (r_1 + r_2)} \binom{k-1}{r_1 + r_2 - 1}, \end{aligned}$$

where we explain the combinatorics identity as followed: $\binom{k-1}{r_1+r_2-1}$ is the number of ways we can write a sum of $r_1 + r_2$ positive integer summands, adding up to k. But that is equivalent to considering the number of ways we can write the first r_1 summands adding up to i and the last r_2 summands adding up to $k - i$, where i can be r_1 up to $k - r_2$ (since the least the last r_2 summand can sum up to be is r_2).