Functions of continuous RV

Math 477

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1 Monotone function of a continuous RV

Theorem 1.1. *Let X be a continuous RV with pdf $f_X(x)$. Suppose $g(x)$ is a strictly monotonic and differentiable function of x. Then $Y = g(X)$ is a continuous RV with density*

$$
\begin{aligned}
f_Y(y) &= f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| \text{ if } y = g(x) \text{ for some } x \\
&= 0 \text{ otherwise.}
\end{aligned}
$$

Proof.

Suppose g is increasing.

$$P(Y \leq y) = P(g(X) \leq y) = P(X \leq g^{-1}(y)) = F_X(g^{-1}(y)).$$

Differentiating yields

$$f_Y(y) = f_X(g^{-1}(y)) \frac{d}{dy} g^{-1}(y).$$

The case when g is decreasing can be proved similarly.

1.1 Examples

Example 1.2. *Let X be a continuous non-negative RV with density function f and let $Y = X^n$. Then*

$$f_Y(y) = \frac{1}{n} y^{1/n-1} f_X(y^{1/n}).$$

Example 1.3. *The log normal distribution Let X be a Normal(μ, σ^2) distribution. Then $Y = e^X$ is saidto have a lognormal distribution with parameters μ, σ^2. Y has density*

$$f_Y(y) = \frac{1}{\sqrt{2\pi\sigma^2}y} e^{-\frac{(\log(y)-\mu)^2}{2\sigma^2}}.$$

2 Non-monotone function of continuous RV

Example 2.1. *Let X have Uniform $[-1, 1]$ distribution. What is the pdf of $Y = X^2$?*

Ans: For $0 \le y \le 1$

$$P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \sqrt{y}.$$

Therefore, $f_Y(y) = \frac{1}{2\sqrt{y}}$.

Remark: The function $g(x) = x^2$ is NOT monotone on $[-1, 1]$. However, noting that for $x < 0$, the inverse of x^2 would be $-\sqrt{x}$ and

$$|d/dx(-\sqrt{x})| = |d/dx\sqrt{x}| = \frac{1}{2\sqrt{x}},$$

we can try to apply the Theorem (1.1) to see what happens. Specifically, we would have

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{d}{dy} g^{-1}(y) \right| = \frac{1}{2} \frac{d}{dy} \sqrt{y} = \frac{1}{4\sqrt{y}},$$

and this is incorrect.

Note that we can try to correct the situation by saying since there are "two" inverse functions of x^2 on the two intervals $[-1, 0]$ and $[0, 1]$ we should add over these to get

$$f_Y(y) = \frac{1}{4\sqrt{y}} + \frac{1}{4\sqrt{y}} = \frac{1}{2\sqrt{y}},$$

which now agrees with our previous result. But note how this is a not straightforward argument.

The above correction may rely on the fact that f_X is symmetric over 0. Let's try a non-symmetric example

Example 2.2. *Let X have Uniform $[-2, 1]$ distribution. What is the pdf of $Y = X^2$?*

Ans: We need to distinguish between $1 \le y \le 4$ and $0 \le y \le 1$. For $0 \le y \le 1$

$$P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le \sqrt{y}) = \frac{2\sqrt{y}}{3}.$$

For $1 \le y \le 4$

$$P(Y \le y) = P(X^2 \le y) = P(-\sqrt{y} \le X \le 1) = \frac{1+\sqrt{y}}{3}.$$

Thus we see that the pdf is

$$\begin{aligned} f_Y(y) &= \frac{1}{3\sqrt{y}}, 0 \le y < 1 \\ &= \frac{1}{6\sqrt{y}}, 1 \le y \le 4. \end{aligned}$$

Again, one can apply the theorem (1.1) with care. On the interval [-2,-1], there is "one" inverse function for $g(x) = x^2$ and on the interval [-1,1] there are "two" inverse functions for $g(x) = x^2$. Thus we need to add for those values of y that have the inverse in the interval [-1,1], namely $0 \le y \le 1$ giving

$$f_Y(y) = \frac{1}{3}\frac{1}{2\sqrt{y}} + \frac{1}{3}\frac{1}{2\sqrt{y}} = \frac{1}{3\sqrt{y}}.$$

And for those values of y that have inverse in the interval $[-2,-1]$, namely $2 \le y \le 4$, we do not have to add, giving

$$f_Y(y) = \frac{1}{3}\frac{1}{2\sqrt{y}} = \frac{1}{6\sqrt{y}}.$$

Let's look at a last example where the inverses are different over different intervals

Example 2.3. *Let X have Uniform $[-1,1]$ distribution. Define*

$$\begin{aligned} g(x) &= x^2, 0 \le x \le 1 \\ &= x^4, -1 \le x < 0. \end{aligned}$$

What is the pdf of $Y = g(X)$?

Ans: For $0 \le y \le 1$

$$P(Y \le y) = P(-y^{1/4} \le X \le \sqrt{y}) = \frac{\sqrt{y} + y^{1/4}}{2}.$$

Thus,

$$\begin{aligned} f_Y(y) &= \frac{1}{4\sqrt{y}} + \frac{1}{8y^{3/4}}, 0 \leq y \leq 1 \\ &= 0 \text{ otherwise .} \end{aligned}$$

Again, note that $g(x)$ have "two" inverses, depending on $x \in [-1,0]$ or $x \in [0,1]$. Therefore, we need to add over these. Thus applying Theorem (1.1) gives

$$f_Y(y) = \frac{1}{2}|d/dy - y^{1/4}| + \frac{1}{2}|d/dy\sqrt{y}| = \frac{1}{4\sqrt{y}} + \frac{1}{8y^{3/4}},$$

agreeing with our previous result.

3 Translating and scaling of continuous RV

A particular function of interest for us, which is always monotonic is $f(x) = \frac{x-a}{b}$. You should verify the following results:

1. If X has a Uniform [a,b] distribution then $Y = \frac{X-\mu}{\sigma}$ has Uniform $[\frac{a-\mu}{\sigma}, \frac{b-\mu}{\sigma}]$ distribution.

2. If X has Normal(μ, σ^2) distribution then $Y = \frac{X-a}{b}$ has Normal$(\mu - a, \left(\frac{\sigma}{b}\right)^2)$ distribution.

3. If X has Exponential(λ) distribution. Then $Y = \frac{X}{\sigma}$ have Exponential$(\frac{\lambda}{\sigma})$ distribution.

4. If X has Exponential(λ) distribution. Then $Y = \frac{X-\mu}{\sigma}$ does NOT have Exponential distribution, simply because the support of Y no longer is $[0,\infty)$. However, you can still say that X has some "exponential type" distribution with support on $[-\mu,\infty)$ and you can work out its density.