The Uniform, Exponential, Normal and Gamma RVs

Math 477

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1 The Uniform

Definition 1.1. *X is said to have a Uniform $[a, b]$ distribution if X has the pdf*

$$\begin{aligned} f_X(x) &= \frac{1}{b-a}, a \leq x \leq b \\ &= 0 \textit{ otherwise.} \end{aligned}$$

If X has Uniform $[a, b]$ distribution then

$$E(X) = \int_a^b \frac{x}{b-a} dx = \frac{a+b}{2}.$$

$$E(X^2) = \int_a^b \frac{x^2}{(} b-a) dx = \frac{b^2+ab+a^2}{3}.$$

Therefore

$$Var(X) = \frac{b^2+ab+a^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{(b-a)^2}{12}.$$

2 The Exponential

Definition 2.1. *X is said to have a Exponential λ distribution if X has the pdf*

$$\begin{aligned} f_X(x) &= \lambda e^{-\lambda x}, 0 \leq x \\ &= 0 \textit{ otherwise.} \end{aligned}$$

If X has Exp(λ) distribution then

$$E(X) = \int_0^\infty \lambda x e^{-\lambda x} dx = \frac{1}{\lambda},$$

by integration by parts.

$$E(X^2) = \int_0^\infty \lambda x^2 e^{-\lambda x} dx = \frac{2}{\lambda^2},$$

again by integration by parts.

Thus

$$Var(X) = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}.$$

Example 2.2. *Buses arrive at the Hill center stop at 15-minute intervals, starting at 7 am. If a student arrives at the stop uniformly between 7 - 7:30 am. What is the probability that he waits*

a. Less than 5 minutes for a bus?

b. More than 10 minutes for a bus?

Ans:

a. This is equivalent to him arriving in the interval [7:10 - 7:15], [7:25-7:30]. Thus the probability is $10/30 = 1/3$.

b. This is equivalent to him arriving in the interval [7:00 - 7:05], [7:15-7:20]. Thus the probability is also $10/30 = 1/3$.

2.1 The memoryless property of the Exponential

Example 2.3. *Suppose the number of miles that a car can run befores its battery wears out is exponentially distributed with an average of 10 k miles. John's car has covered 4 k miles since the last battery change. He plans to take another 5k mile trip. What is the probability that he will be able to complete the trip without having to replace the battery?*

Ans: Let X be the life time of the battery in thousand of miles. Then X has Exponential(1/10) distribution. We want to compute $P(X > 9|X > 4)$. Then

$$\begin{aligned} P(X > 9|X > 4) = \frac{P(X > 9)}{P(X > 4)} &= \frac{\int_9^\infty \frac{1}{10} e^{-\frac{1}{10}x} dx}{\int_4^\infty \frac{1}{10} e^{-\frac{1}{10}x} dx} \\ &= \frac{e^{-9/10}}{e^{-4/10}} = e^{-5/10}. \end{aligned}$$

Observe that the above answer is exactly as if we computed

$$P(X > 5) = \int_5^\infty \frac{1}{10} e^{-\frac{1}{10}x} dx = e^{-5/10}.$$

In fact, you can see that this is a general property of the Exponential(λ): for $t > s$

$$P(X > t | X > s) = P(X > t - s).$$

We refer to this as the memoryless property. The interpretation is that the machine or whatever instrument that has its lifetime distributed as an exponential RV does not "remember" that it has survived an interval of length s; if we want to compute its chance of survival beyond an interval of length t, given that it has survived an interval of length s.

The following converse is also true: If X is a continuous random variable such that X satisfies the memoryless property then X must have an Exponential distribution.

2.2 Exponential RV as model for waiting time of Poisson event arrival

The Exponential RV also arises as model for the inter-arrival time of events that has arrival distribution according to a Poisson process:

Example 2.4. *Earth quake occurs in California with rate 2 per week.*

a. What is the probability that there will be no earth quake during the next month?

b. Let T be the number of weeks until the next earthquake (T is a continuous random variable). What is the distribution of T?

Ans: a. The number earth quake within next month is Possion (8). Thus

$$P(X = 0) = e^{-8}.$$

b. If $T > t$ it means that there is no earthquake within t weeks. That is $P(T > t) = e^{-2t}$. We also have $P(T \leq t) = 1 - e^{-\lambda t}$.

3 The Normal RV

Definition 3.1. *X is said to have a Normal(μ, σ^2) distribution if X has the pdf*

$$f_X(x) \quad = \quad \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}, -\infty < x < \infty.$$

If X has Normal(μ, σ^2) distribution then

$$E(X) = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mu,$$

by substitution.

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx = \mu^2 + \sigma^2.$$

by integration by parts and substitution.

Thus

$$Var(X) = \mu^2 + \sigma^2 - \mu^2 = \sigma^2.$$

3.1 The cumulative distribution of a Normal

Observe that for a Normal(μ, σ^2)

$$P(X \leq x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(u-\mu)^2}{2\sigma^2}} du$$

does not have an explicit formula. One can still approximate this value by numerical method, which gives rise to the Normal table. The quantity $P(X \leq x)$ is the c.d.f. of X, of course, and we denote it by $\Phi(x)$.

We observe the following properties of Φ:

a. $\Phi(x) = 1 - \Phi(-x)$. Thus one only needs to give $\Phi(x)$ for $x \geq 0$.

b. $\Phi(0) = 1/2$.

Note: Some normal table does not give $\Phi(x)$ however. They give $P(0 \leq X \leq x)$ for $x \geq 0$ or $\Phi(x) - 1/2$. One should check the normal table to see what value they give before using it.

3.2 The Normal approximation to the Binomial

Theorem 3.2. *DeMoivre - Laplace Let X be a Binomial(n,p). Then for n large:*

$$P(a \leq \frac{X - np}{\sqrt{np(1-p)}} \leq b) \approx \Phi(b) - \Phi(a).$$

3.3 The continuity correction

The above Theorem allow us to compute, for X having Binomial (n,p) distribution $P(\tilde{a} \leq X \leq \tilde{b})$, where a, b are real numbers. More specifically:

$$\begin{aligned}\tilde{a} &= a\sqrt{np(1-p)} + np \\ \tilde{b} &= b\sqrt{np(1-p)} + np.\end{aligned}$$

But since X is a discrete random variables, it only takes integer values. Thus we most often start out with expression of the form $P(m \leq X \leq n)$ where m, n are integers. Because the point mass probability $P(X = m)$ and $P(X = n)$ are included in the expression $P(m \leq X \leq n)$, we do not want to miss it in the transformation into the form $P(\tilde{a} \leq X \leq b)$. Thus we choose $\tilde{a} = m - 1/2$ and $\tilde{b} = n + 1/2$. This is called *the continuity correction.*

4 The Gamma distribution

Definition 4.1. *X is said to have a Gamma(α, λ) distribution if X has the pdf*

$$\begin{aligned}f_X(x) &= \frac{\lambda e^{-\lambda x}(\lambda x)^{\alpha-1}}{\Gamma(\alpha)}, x \geq 0 \\ &= 0 \textit{ otherwise }.\end{aligned}$$

Where $\Gamma(\alpha)$, the gamma function, is defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-y} y^{\alpha-1} dy.$$

4.1 Properties of the Gamma function

By integration by parts, we have

a. $\Gamma(\alpha) = (\alpha - 1)\Gamma(\alpha - 1)$.

b. $\Gamma(1) = 1$.

From which it follows that $\Gamma(n) = (n-1)!$ for n a positive integer.

4.2 Gamma distribution as model for waiting time for n events

Suppose events arrive according to a Poisson process distribution. Then the amount of time one has to wait until a total of n events has occured will be a Gamma(n, λ)

distribution. More specifically, let T_n be the time that the nth event occurs. Then

$$P(T_n \leq t) = P(N(t) \geq n) = \sum_{k=n}^{\infty} \frac{e^{-\lambda t}(\lambda t)^k}{k!}.$$

Differentiating yields:

$$\begin{aligned} f_{T_n}(t) &= \sum_{k=n}^{\infty} \frac{\lambda e^{-\lambda t}(\lambda t)^{k-1}}{(k-1)!} - \sum_{k=n}^{\infty} \frac{\lambda e^{-\lambda t}(\lambda t)^k}{k!} \\ &= \frac{\lambda e^{-\lambda t}(\lambda t)^{n-1}}{(n-1)!}. \end{aligned}$$

4.3 Gamma as sum of the Exponentials

We see that if $\alpha = n$ then the Gamma(n, λ) random variable represents the arrival time of the nth event with arrival rate λ. This suggests that if we have $X_1, X_2, \cdots, X_n$ independent Exponential(λ) then $X_1+X_2+\cdots X_n$ has distributon Gamma(n, λ). This turns out to be correct, and we'll show the proof in Chapter 6 when we cover the sum of independent RVs.

The importance of the above interpretation is that if Y has Gamma(n, λ) then we can deduce

$$\begin{aligned} E(Y) &= \sum_i E(X_i) = \frac{n}{\lambda} \\ \mathrm{Var}(Y) &= \sum_i \mathrm{Var}(X_i) = \frac{n}{\lambda^2}. \end{aligned}$$

What about the case when Y has Gamma(α, λ), where α is not an integer? Even though we lose the interpretation of Y as the sum of the Exponentials, the intuition about the sum of expectation and variance still applies, to give us

$$\begin{aligned} E(Y) &= \frac{\alpha}{\lambda} \\ \mathrm{Var}(Y) &= \frac{\alpha}{\lambda^2}. \end{aligned}$$

The rigorous proof, of course is by integration. We won't present it here.