

# Combinatorics

Math 477

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## 1 Introduction - The need to count

### 1.1 About combinatorics

Combinatorics (at least at the elementary level) is the study of counting. You may wonder why we need to study something that any elementary student can perform. The answer is in the method. Basically how did we learn to count before? Roughly by using our fingers, or by listing everything out and start going 1,2,3 etc. This clearly is an inefficient way to count. And all of us know in some situation we can be more efficient than that. For example, if we lay oranges in a rectangle formation, with 10 oranges on one side and 4 oranges on the other side, then we know we have  $10 \times 4 = 40$  oranges total. Coincidentally, the Field's medalist Manjul Bhargava of this year (2014) considered a similar question when he was 8: how many oranges are there in a pyramid formation with length  $n$  on each side? Hint: He didn't solve it using his fingers. You can find out the answer and read more about his story here: <http://www.simonsfoundation.org/quanta/20140812-the-musical-magical-number-theorist/>

It turns out that there are many situations that simply listing things out is not possible (or desirable). This chapter on combinatorics will show us basic techniques that can be applied to basic situations where counting can be done more efficiently than listing. You can say that more advanced combinatorics essentially deals with the same question, just in much much more subtle and complicated scenarios.

## 1.2 Connection with probability

We will discuss the notion of probability in a systematic way in the next chapter. For now, let's just consider an example. Suppose you roll a 2 dice and ask how likely it is to get a sum of 11 versus getting a sum of 7. Now if you give it some thoughts, you'll see that getting a sum of 7 is easier. We say the probability of rolling a sum of 7 is higher than rolling a sum of 11. Why is this? Basically there are *more* ways of rolling a 7 (for example: (1,6), (2,5), (3,4) and their reverses) than of rolling an 11 (just (5,6) and (6,5)). Notice that we counted the possibilities there by listing. But again, we can imagine situations where we need more sophisticated methods of counting, to decide likelihood of events. This is exactly how combinatorics ties into probability, at least for the introductory chapters. We use it as a *technique* to help us decide probabilities.

## 2 The basic principle of counting

### 2.1 The principle

Suppose there are  $k$  experiments to be performed, the  $i$ th experiment having  $n_i$  possible outcomes, then together there are  $n_1 n_2 \cdots n_k$  possible outcomes.

### 2.2 Discussion

This principle is used in many places. A (rather artificial) example is if you go to a restaurant and there are 3 choices for appetizers, 5 choices for entrees and 4 choices for desserts then together you can form  $3 \times 4 \times 5 = 60$  variations for a dinner. As far as probability is concerned, there are many times when we repeat an experiment, for example rolling a die twice (equivalent to rolling 2 similar dice). In this case we have a total of  $6 \times 6 = 6^2$  possible outcomes, for example (1, 2), (2, 1), (2, 3), (3, 2) etc. Similarly toss a coin 3 times, we have a total of  $2^3 = 8$  possible outcomes, for example *HHT*, *HTH*, *TTT*, etc. It is good to know how many outcomes we have in total, because that will be the basis for forming the probability of an event (for example, the probability of rolling an 11 when rolling 2 dice is  $2/36$ , because there are only 2 ways to get an 11 out of 36 possible outcomes).

Note that there are cases when this principle does not apply. For example, a deck of cards has 52 cards total. So when we draw a hand, there are 52 possibilities (ace

of heart, 10 of club etc). But when we draw 2 cards, it *does not* follow that there are  $52^2$  possibilities for our 2 cards, since the experiment (of drawing 1 card from the deck) is *not repeated*. This is easy to see, because among the 2 cards drawing from the same deck, we can never get the outcome of 2 aces of heart, for example; but this outcome is definitely included among the  $52^2$  possibilities. Note, however, that if we draw from 2 *different* decks of card, or equivalently, drawing 1 card, put it back to the deck and draw another card, then the principle applies, and we do have  $52^2$  possibilities.

Similarly, later on we will have examples about drawing balls from an urn. A typical example is an urn having 3 balls of blue and 4 balls of red. There are 2 different ways we draw balls from the urn: with or without replacement. *Drawing with replacement* means we pick a ball, drop it back to the urn and then draw the next ball. This is the same as repeating an experiment. *Drawing without replacement* means we just pick a couple of balls from the urn *at the same time*. In this way the experiment (of picking 1 ball from the urn) is not repeated. To count the number of total outcomes in this case needs other principles, permutation and combination (depending whether order matters or not), which we'll cover next in this chapter.

### 3 Permutation

Consider the letters  $ABC$ . There are several ways we can re-arrange them:  $BCA, ACB$  etc. We can each of this arrangement *a permutation*. So how many ways in total can we re-arrange the letters  $ABC$ ? After some thinking, you'll see that there are 6 ways total. This comes from the fact that there are 3 choices for the first position, 2 choices for the second position and 1 choice for the third position so from the principle above, there are  $3 \times 2 \times 1 = 6$  permutations total.

*In general, we conclude that there are  $n! = n(n-1)(n-2)\cdots 2 \cdot 1$  permutations of  $n$  objects.*

The permutation, together with the basic principle, are surprisingly all we need to deal with many interesting scenarios in counting. Some basic examples are as followed.

Ex1. Suppose we have 3 math books and 5 physics books and 6 chemistry books. How many ways can we re-arrange them so that all the books of the same subject are together?

Ans: There are  $3!$  arrangements for the math books,  $5!$  for physics and  $6!$  for

chemistry. So if we go with MPC (M = math, P = physics, C = chemistry) arrangement, there are  $3!5!6!$  possible permutations. But this is not the only possible arrangement for the subjects. One can go with CPM for example. And it's easy to see there are  $3!$  arrangements for the subject themselves. Since each possible arrangement of the subjects has  $3!5!6!$  possible permutations (of the books within each subject), in total we have  $3!3!5!6!$  possible permutations.

**Remark 3.1.** *Implicit in this problem is the assumption that all math books are different, similarly for physics and chemistry. If we have 3 identical physics books among the 5, for example, then the counting has to be modified. In this case we say all the books within a subject are **distinguishable**. In some context where we want to emphasize this fact (or the converse, that they are identical), we say the objects are **distinguishable** (or the objects are **indistinguishable**). See the example at Section (6). Another simpler way to say is the objects are **different** (or **identical**).*

Ex2. How many arrangements can be made out of the letters  $CCCAABAAB$  ?

Ans: It is good to try a few possible arrangements and see what we can get. These are:  $CAAAABCC$ ,  $BBAAAACC$ ,  $AACCCAABB$ , for example. At first it may not be clear how to proceed with this problem, since there's not clearly a "group" that the A's can fall into (sometimes they're split into 2 groups), similarly for the B's and C's. Another is all the  $A, B, C$ 's are the same.

In counting, it is sometimes a helpful trick to *treat identical objects as if they are different*, and then account for the fact that they are actually the same. For example here, we can suppose what we have is  $A_1A_2A_3A_4B_1B_2C_1C_2C_3$ . Then all letters are different, and there are  $9!$  permutations total. But in doing this, we obviously overcount. And the overcount comes from the permutations among the  $A, B, C$ 's, since they are not really different. For example, in the arrangement  $AAAABBBCCC$ , the 2 permutations  $A_2A_1A_4A_3B_2B_1C_2C_1C_3$  and  $A_1A_2A_4A_3B_1B_2C_1C_2C_3$  are the same. It is easy to see that for each particular arrangement like this, there are  $4!3!2!$  permutations of letters among each group (just like the book example). And by the basic principle, if  $x$  is the number of arrangements of interest, then  $x4!3!2! = 9!$ . Thus  $x = \frac{9!}{4!3!2!}$ .

This example leads to the following conclusion: *There are  $\frac{n!}{n_1!n_2!\dots n_r!}$  different permutations of  $n$  objects, of which  $n_1$  are alike,  $n_2$  are alike,  $\dots$ ,  $n_r$  are alike.*

## 4 Combination

A common question in counting is how many ways can we pick  $r$  objects out of  $n$  objects? For example, suppose we have 5 letter  $ABCDE$ . How many ways can we pick 3 letters out of these 5? Again, it's help ful to try a few examples. We can pick  $ADE$ ,  $ABC$ ,  $CDE$  etc. But note that in this way,  $ADE$  and  $DEA$  are the same to us, because we're not interested in how many permutations we can form out of the group we pick. Now we can answer the question, since suppose we pick the letters sequentially, *without replacement*, then there are 5 choices for the 1st pick, 4 choices for the 2nd and 3 choices for the 3rd. Thus there are  $5 \cdot 4 \cdot 3$  ways total we can pick 3 letters out of 5. BUT in this way we overcount, since  $ADE$  and  $DEA$  are counted as different here. So to correct for this, let  $x$  be the number of ways we are interested in. Then for each way, there are  $3!$  permutations of objects within it. That is  $x3! = 5 \cdot 4 \cdot 3$ . Thus  $x = \frac{5 \cdot 4 \cdot 3}{3!} = \frac{5!}{3!2!}$ .

Note: You can easily see that then the number of ways we pick 2 letters out of 5 is also  $\frac{5!}{3!2!}$ . This is clear, since if we pick 3 out of 5, then the remaining is 2. Thus in general, the number of ways we pick  $r$  out of  $n$  must be equal to the number of ways we pick  $n - r$  out of  $n$ .

We denote  $\binom{n}{r} = \frac{n!}{(n-r)!r!}$ , as the number of ways we can pick  $r$  objects out of  $n$  objects, also referred to as the number of possible *combinations* of  $n$  objects taken  $r$  at a time.

### 4.1 A generalization

Picking  $r$  out of  $n$  can be thought of as dividing  $n$  objects into 2 groups, 1 consisting of  $r$  and the other  $n - r$  objects. Sometimes, we want to divide  $n$  objects in more than 2 groups. Consider the following example.

Ex1: Suppose we have 9 letters. How many ways can we divide them into 3 groups, consisting of 4,3 and 2 letters each?

Ans: There are  $\binom{9}{4}$  ways of picking the first group of 4 letters, leaving 5 letters left, from which there are  $\binom{5}{3}$  ways of picking 3 letters. Thus by the first principle, there are

$$\binom{9}{4} \binom{5}{3} = \frac{9!}{5!4!} \frac{5!}{3!2!} = \frac{9!}{4!3!2!}$$

ways total.

Ex2: How many different signals, each consisting of 9 flags hung in a line, can be

made from a set of 4 white flags, 3 red flags and 2 blue flags, if all flags all the same colors are identical?

Ans: This question actually is a permutation question. From the discussion above, we see that there are  $\frac{9!}{4!3!2!}$  different signals, same answer as Example 1 ! However, I would recommend you to go through the reasoning in the permutation section to see how we derive this answer, and compare it with the way we reason for the combination number. The point is these same answers have different interpretations, depending on the context. In the permutation question, we *already have 3 groups, and we permute members of the groups among each other*. In the combination question, we *form the 3 groups out of the 9 objects*.

Since the answers are the same, actually one can go from one interpretation to the other. To go from Ex1 to Ex2, you can think of 9 positions, and we simply choose 3 groups of 4,3,2 positions each. To each group we simply put the flags of the corresponding color (for example, suppose we pick position 3, 5,6, 8 for the first group, then we will put the white flags in those positions). Now since it's a combination, the *permutation of the positions within each group* does not matter. This is exactly what the *same color* of the flags mean. To go from Ex2 to Ex1, you can think of the positions of each group of flags as marking the positions of the particular objects we pick out among the 9 objects. Then again, since the permutation of the positions of the flags of the same color does not result in a new arrangement, this is exactly what we want when we think of a picking a group of objects as a combination.

The bottom line of these 2 examples is there are many ways to approach a problem of counting. I only recommend that when we deal with question in counting, it is good to keep the context in mind, and be clear (have a mental picture) about the procedure we're employing.

## 5 Binomial and multinomial coefficients

### 5.1 The binomial theorem

We have

$$(x + y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}.$$

*Proof.* The proof of the binomial theorem lies in the expansion of the expression  $(x + y)^n = (x + y)(x + y) \cdots (x + y)$  ( $n$  times). First, you see that the expansion

will consist of sum of terms of the forms  $x^k y^j$  where  $k, j$  are integers. But we can do better. Observe that there are  $n$   $(x + y)$  terms, each containing both  $x$  and  $y$ . When we form a particular  $x^k y^j$  term, what we do is we *pick the  $(x + y)$  terms* where we would select the  $x$ 's from (and by the expression, we need to pick  $k$  such terms) and select the  $y$ 's out of the remaining  $(x + y)$  terms (also by the expression, there must be  $j$  remaining terms). Thus the terms  $x^k y^j$  must actually be of the form  $x^k y^{n-k}$  (if you pick  $k$   $x$ 's then you have to pick  $n - k$   $y$ 's from the rest) and there are  $\binom{n}{k}$  ways to pick them ( out of the  $n$   $(x + y)$  terms, we choose  $k$  terms to select the  $x$ 's from, and the  $y$ 's come from the rest).

The number  $\binom{n}{k}$  is referred to as the binomial coefficient. It expresses how many ways one can pick  $k$  objects out of  $n$  objects as we have discussed.

## 5.2 The multinomial theorem

We have

$$(x_1 + x_2 + \cdots + x_r)^n = \sum_{(n_1, \dots, n_r): n_1 + \dots + n_r = n} \binom{n}{n_1, n_2, \dots, n_r} x_1^{n_1} x_2^{n_2} \cdots x_r^{n_r}.$$

The sum is understood to be taken over all tuples (that is, vectors)  $(n_1, \dots, n_r)$  of integers such that their sum is equal to  $n$ . The proof is similar to the binomial theorem, except now in the expansion we need to pick  $n_1$  of  $(x_1 + x_2 + \cdots + x_r)$  to select  $x_1$  from,  $n_2$  of such terms to select  $x_2$  from etc.

The number  $\binom{n}{n_1, n_2, \dots, n_r}$  is referred to as the multinomial coefficient. It represents how many ways we can form groups of  $n_1, n_2, \dots, n_r$  objects out of  $n$  objects as we have discussed.

## 6 The risk of overcounting

In doing combinatorics problems, one can easily make the mistake of overcounting without realizing it, thus leading to the wrong answer to the problem. It is important to be aware of overcounting so that we can minimize the chance of making errors. The following example will demonstrate.

Ex: Consider a set of  $n$  antennas of which  $m$  are defective and  $n - m$  are functional, assuming all defectives and all functionals are indistinguishable. How many linear orderings (that is how many ways can we line them up) so that no two defectives are consecutive?

Ans: At first it is not obvious how to approach this problem. A (possibly) good approach is to look at the complimentary problem: find out how many linear orderings in which there are *at least* two consecutive defectives. Let's say the answer is  $x$ . Since there are  $\binom{n}{m}$  linear arrangements total, the answer to the original question would be  $\binom{n}{m} - x$ .

One way (turns out to be *incorrect*) to reason is: there are  $(n - 1)$  consecutive positions in a linear ordering (e.g. positions (1,2), (2,3), (3,4) etc.). We will put the 2 defective antennas into these positions. After we pick a pair of consecutive positions, say (3,4), then we just have to put the rest of the antennas into the remaining positions. There are  $\binom{n-2}{m-2}$  ways of doing so (since there are  $n - 2$  positions left and we pick  $m - 2$  positions for the remaining defective antennas). Thus the number of linear orderings in which there are *at least* two consecutive defectives is

$$(n - 1) \binom{n - 2}{m - 2}.$$

This turns out to be *incorrect*. The reason is we *overcount*, meaning there are particular arrangements that we count more than once. For example, it is easy to see that there are overlapping arrangements when we pick the position (1,2) for 2 defectives and arrange the rest into places, versus picking position (2,3) for 2 defectives and arrange the rest into places (draw such an arrangement for yourself). One can still make this approach work by taking care of this problem of overcounting, but it is easy to see how it quickly gets complicated.

The right way to approach the problem is to realize to make none of the defective consecutive, one need at least a functional in between any of them. This means if we line up  $n - m$  functionals, then we must put the  $m$  defectives in the *gap* between 2 consecutive functionals, or the 2 positions before and after all the functionals:

$$\wedge 1 \wedge 1 \wedge 1 \cdots 1 \wedge 1 \wedge$$

(the 1's represent the functionals, we put the defectives where the  $\wedge$  are)

There are  $n - m + 1$  such positions. Thus there are  $\binom{n-m+1}{m}$  arrangements total.