Math 477 **Name (Print):** ____________________

Fall 2014

Final exam

12/19/2014

This exam contains 9 pages (including this cover page) and 10 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	25	
2	20	
3	30	
4	20	
5	20	
6	20	
7	20	
8	15	
9	15	
10	20	
Total:	205	

1. Due to negative ratings, the group of 10 friends fill not be featured in the Math 477 final exam. However, before they say goodbye, they still go to a restaurant one last time ! The two groups of boys and girls arrive at the restaurant independently and uniformly from 9:00 am to 11:00 am.

   (a) (20 points) What is the expectation of their wait time for each other?

   Ans:

   Let X, Y have joint Uniform $[(0,2) \times (0,2)]$ distribution.

$$\begin{aligned} E|X-Y| &= \int_0^2 \int_0^2 \frac{1}{4}|x-y|dxdy \\ &= 1/2 \int_0^2 \int_y^2 (x-y)dxdy \\ &= 1/2 \int_0^2 (2-y)^2 dy \\ &= 2/3. \end{aligned}$$

   (b) (5 points) (Extra credit) What are the names of the 10 friends? (You get 0.5 point per name up to a maximum of 5 points).

   Tom, Tim, Tony, Todd, Ted and Jane, June, Jill, Judy, Joan.

2. (20 points) In the mood of holiday spirit, the instructor of Math 477, let us call him Mr. T (no relation to the other famous Mr. T), decorates his Christmas tree. He has a 2 boxes of ornaments, each containing 10 ornaments of red and blue color, respectively. Each time he hangs an ornament, he is equally likely to select an ornament from the red box or the blue box. While singing the Christmas Carol, Mr. T discovers that his blue ornament box is empty. What is the probability that there are 3 remaining ornaments in the red box?

   Ans: The probability that he picks a red or a blue ornament is 1/2 each. We call each time he picks a blue ornament a success and a red ornament a failure. Then this is asking for the probability of having 11 successes (counting the last one that he checks to know the blue ornament box is empty) in 18 trials, with the LAST one being a success. In other words, if we let X be a Negative Binomial (18, 1/2) then this is

$$P(X=11) = \binom{17}{10}(1/2)^{18}.$$

3. Let X, Y have joint density

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{2}, 0 \leq x \leq y \leq 1 \\ &= 0, \text{ otherwise }. \end{aligned}$$

(a) (10 points) Find $Cov(X,Y)$.

$$\begin{aligned} E(XY) &= \int_0^1 \int_0^y \frac{xy}{2} dx dy = \int_0^1 \frac{y^3}{4} dy = \frac{1}{16}; \\ E(X) &= \int_0^1 \int_0^y \frac{x}{2} dx dy = \int_0^1 \frac{y^2}{4} = \frac{1}{12}; \\ E(Y) &= \int_0^1 \int_0^y \frac{y}{2} dx dy = \int_0^1 \frac{y^2}{2} = \frac{1}{6}. \end{aligned}$$

Thus

$$Cov(X,Y) = E(XY) - E(X)E(Y) = 1/16 - 1/72.$$

(b) (10 points) Let Z have Normal(5,10) distribution, Z independent of X. Find

$$Cov(X, 2Y + 3Z).$$

Since Z is independent of X, $Cov(X,Z) = 0$.
Thus

$$Cov(X, 2Y + 3Z) = 2Cov(X,Y) + 3Cov(X,Z) = 1/8 - 1/36.$$

(c) (10 points) Find the moment generating function of $Z = X + Y$.

$$\begin{aligned} M_Z(t) &= E(e^{tZ}) = \int_0^1 \int_0^y e^{t(x+y)} dx dy \\ &= \int_0^1 e^{ty} \frac{e^{ty} - 1}{t} dy \\ &= \frac{1}{t} \left[\frac{e^{2t} - 2e^t}{2t} + \frac{1}{2t} \right] \\ &= \frac{1}{2} \left(\frac{e^t - 1}{t} \right)^2. \end{aligned}$$

4. Let X, Y have joint density

$$\begin{aligned} f_{XY}(x,y) &= \frac{1}{x^2y^2}, x \geq 1, y \geq 1 \\ &= 0, \text{ otherwise }. \end{aligned}$$

(a) (10 points) Find the joint density of $U = XY, V = X/Y$.
Ans:
The inverse map is

$$\begin{aligned} x &= \sqrt{uv} \\ y &= \sqrt{\frac{u}{v}}. \end{aligned}$$

The Jacobian matrix is

$$\begin{bmatrix} y & x \\ \frac{1}{y} & \frac{-x}{y^2} \end{bmatrix}.$$

Its determinant is $-2\frac{x}{y} = -2v$.
Therefore,

$$f_{UV}(u,v) = \frac{1}{u^2}\frac{1}{2v} = \frac{1}{2u^2v},$$

where $u, v \geq 0, uv \geq 1, \frac{u}{v} \geq 1$.

(b) (10 points) Find the marginal density of U.

$$\begin{aligned} f_U(u) &= \int_{\frac{1}{u}}^{u} \frac{1}{2u^2v} dv \\ &= \frac{1}{2u^2}(\log(u) - \log(1/u)) \\ &= \frac{\log(u)}{u^2}, u \geq 1. \end{aligned}$$

Note that this is an actual density because

$$\int_1^\infty \frac{\log(u)}{u^2} du = -\frac{1}{u}\log(u)\Big|_1^\infty + \int_1^\infty \frac{1}{u^2} du = 1.$$

5. Mr. T goes to ballroom dancing with 4 other guys in the Math department. At the ballroom, they meet 5 other ladies. Suppose each guy independently chooses a lady at random to ask her to dance with him, and the probability that she agrees to the proposal is p (and yes, two guys can dance with the same lady, they just have to wait in turn). Let us call any lady who no one dances with a sad lady.

   (a) (10 points) Find the expectation of the number of sad ladies ?

   Let $X_i, i = 1, \cdots 5$ be 1 if the ith lady is a sad lady, 0 otherwise. Then the probability that the first lady dances with the first guy is

   $$\frac{p}{5}.$$

   That is the first guy must ask her with probability 1/5 and she must agree with probability p.

   Therefore

   $$E(X_1) = P(X_1 = 1) = (1 - \frac{p}{5})^5.$$

   Then the expected number of the sad lady is

   $$E(X_1 + X_2 + \cdots X_5) = 5E(X_1) = 5(1 - \frac{p}{5})^5.$$

   (b) (10 points) Answer the same question as part a, if each guy independently chooses 2 ladies at random to ask, and each lady agrees with probability p.

   Set up X_i as above. Now the probability that the first lady dances with the first guy is

   $$p\frac{\binom{4}{1}}{\binom{5}{2}}.$$

   That is the first guy must ask her with probability $\frac{\binom{4}{1}}{\binom{5}{2}}$ and she must agree with probability p.

   Therefore

   $$E(X_1) = P(X_1 = 1) = \left(1 - p\frac{\binom{4}{1}}{\binom{5}{2}}\right)^5.$$

   Then the expected number of the sad lady is

   $$E(X_1 + X_2 + \cdots X_5) = 5E(X_1) = 5\left(1 - p\frac{\binom{4}{1}}{\binom{5}{2}}\right)^5.$$

6. (20 points) The time X that a Math 477 student takes to finish the first midterm is an Exponential(1/2) random variable. There are 25 students in the class, and suppose the time it takes for each one to finish are independent. What is the approximate probability that the average time the students take will exceed 2.5 hours?

$$\begin{aligned} E(\frac{\sum_i X_i}{25}) &= 2 \\ Var(\frac{\sum_i X_i}{25}) &= \frac{4}{25}. \end{aligned}$$

Thus

$$P(\frac{\sum_i X_i}{25} \geq 2.5) \approx P(Z \geq \frac{(2.5-2)\times 5}{2}) = P(Z \geq \frac{2.5}{2}).$$

7. A Math 477 student asks Mr. T for a recommendation letter for admission to Harvard. Suppose the probability that she will get admitted is .8 if she receives a strong letter, .4 if she receives an average letter and .1 if she receives a weak letter. Also suppose that the probabilities that the recommendation will be strong, average and weak are .7, .2 and .1 respectively.

   (a) (10 points) What is the probability that she will be admitted to Harvard?
   Let E be the event that she gest admitted, L_W, L_A, L_S be the event that she gets a weak, average and stronge letter respectively.
   Then

   $$P(E) = P(E|L_W)P(L_W) + P(E|L_A)P(L_A) + P(E|L_S)P(L_S) = (.7)(.8) + (.2)(.4) + (.1)(.1).$$

   (b) (10 points) The poor student found out she did not get admitted to Harvard. What is the conditional probability, based on this event, that she received a weak recommendation letter?

$$\begin{aligned} P(L_W|E^c) &= \frac{P(L_W E^c)}{P(E^c)} \\ &= \frac{P(E^c|L_W)P(L_W)}{1-P(E)} \\ &= \frac{(.9)(.1)}{1-[(.7)(.8)+(.2)(.4)+(.1)(.1)]}. \end{aligned}$$

8. Mr. T and 4 other friends from the department, Mr. G, I , F and Ms. Z go to the movie theatre to check out Interstellar. At the theatre they get in line at a random order for the tickets. Assuming each order is equally likely. What is the probability that

   (a) (5 points) There is exactly 1 person between T and Z?

   There are 3 ways to choose the person to be between T and Z. Let's call that person A for anonymous. Within the group of T, A, Z there are 2 ways to re-arrange so that A stays in the middle: T, A, Z and Z, A, T. There are 3! ways permute the group with 2 other people outside the group. Therefore the answer is $\frac{3\times2\times3!}{120} = \frac{36}{120}$.

   (b) (5 points) There are exactly two people between T and Z?

   There are 3 ways to choose 2 people between T and Z. Let's call them A_1, A_2. Within the group of T, A_1, A_2, Z there are 2×2 ways to re-arrange so that A_1, A_2 stay in the middle. There are 2 ways to permute the group with one person outside the group. Therefore the answer is $\frac{3\times4\times2}{5!} = \frac{24}{120}$.

   (c) (5 points) There are exactly 3 people between T and Z?

   Within the group of T, G, I, F, Z there are $2 \times 3!$ ways to re-arrange so that G, I, F stay in the middle. Therefore the answer is $\frac{2\times6}{120} = \frac{12}{120}$.

   Note: The probability that no person is between T and Z is $\frac{2\times4!}{5!} = \frac{48}{120}$ and you can verify that all the probabilities add up to 1.

9. (15 points) Mr. T is coding a program for Google. He will need to code continuously for at least 50 hours. With the intensity of this kind of coding, Mr. T will set the computer on fire if it is not constantly cooled. Unfortunately, the cooling system for the computer is not very reliable. If the mean time of operation for the cooling system is 5 hours and its standard variation is 2 hours, how many of these components must be in stock so that the probability that Mr. T can code non-stop for the next 50 hours is at least .95?

   Let n be the number of components needed. Clearly the larger the n, the larger the probability that the computer can survive in the next 50 hours. Thus we only need to find the smallest n that would make the probability greater than .95. Let X_i be the life time of each component. Then

$$\begin{aligned} E(\sum_{i=1}^{n} X_i) &= 5n \\ Var(\sum_{i=1}^{n} X_i) &= 4n. \end{aligned}$$

   We have

$$P(\sum_{i=1}^{n} X_i \geq 50) \approx P(Z \geq \frac{50-5n}{2\sqrt{n}}) = .95.$$

   Thus we need $\frac{50-5n}{2\sqrt{n}} \leq -1.65$. That is

$$50 \leq 5n - 3.3\sqrt{n}.$$

   A quick calculator check shows that $n \geq 13$.

10. (20 points) After finish grading for Math 477 final, Mr. T heads to California for the sunshine and vacation. He's driving. Being a poor instructor, his car is not very reliable. It has 5 engines that may break down suddenly. Suppose the car will be able to run as long as at least 3 of its 5 engines are functioning and each engine independently functions for a random amount of time with density

$$f(x) = e^{-x}, x > 0.$$

   Compute the density function of the length of time that the Mr. T's car can survive on the road.

   Let T be the length of time and X_i be the survival time of the ith engine. Note that $P(X_i > t) = P(X_1 > t)$ for all i. So we call $p(t) = P(X_1 > t)$. Then

$$P(T > t) = P(\text{ at least 3 } X_i \ \geq t) = \binom{5}{3} p(t)^3(1-p(t))^2 + \binom{5}{4} p(t)^4(1-p(t)) + p(t)^5.$$

Thus

$$\begin{aligned} f_T(t) &= -d/dtP(T > t) = -\Big[\binom{5}{3} 3p(t)^2 p'(t)(1-p(t))^2 \\ &- \binom{5}{3} p(t)^3 2(1-p(t))p'(t) + \binom{5}{4} 4p(t)^3 p'(t)(1-p(t)) \\ &- \binom{5}{4} p(t)^4 p'(t) + 5p(t)^4 p'(t)\Big]. \end{aligned}$$

where

$$p(t) = \int_t^\infty e^{-x} dx = e^{-t}$$

and

$$p'(t) = -e^{-t}.$$