Math 485
Fall 2013
Final exam
12/17/2013

Name (Print): ____________________

This exam contains 8 pages (including this cover page) plus one page for the normal table and 13 problems. Check to see if any pages are missing. Enter all requested information on the top of this page, and put your initials on the top of every page, in case the pages become separated.

You may use 1 page of note (2 sided) and a scientific calculator on this exam.

You are required to show your work on each problem on this exam. The following rules apply:

- **Organize your work**, in a reasonably neat and coherent way, in the space provided. Work scattered all over the page without a clear ordering will receive very little credit.
- **Mysterious or unsupported answers will not receive full credit**. For example, in question involved the multi-period binomial model, I would like to see how you derive the no arbitrage price, say by displaying the tree with all the nodes filled out if the situation is appropriate.
- **P**artial credits will be given so it is to your advantage to attempt all the questions and not leave any problems blank.
- If you need more space, use the back of the pages; clearly indicate when you have done this.

Problem	Points	Score
1	20	
2	20	
3	15	
4	15	
5	15	
6	15	
7	20	
8	10	
9	10	
10	25	
11	5	
12	10	
13	20	
Total:	200	

1. Let S_t represent the stock price under the risk neutral measure using the Geometric Brownian motion model,i.e.

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

Let V_0 represent the price at time 0 of the cash or nothing option with strike price K and expiration time T, i.e. $V_T = \mathbf{1}_{S_T \geq K}$.

(a) (5 points) Compute V_t, for $0 < t < T$. Ans: See class notes.

(b) (10 points) Compute the Greeks $\Theta(t), \Delta(t), \Gamma(t)$ of this option.
Ans: See class notes.

(c) (5 points) Show that V_t satisfies the Black-Scholes PDE. Ans: See class notes.

2. (a) (10 points) The breakdown time of the Apple Ipad is an exponential(5) random variable. In other words, let X be the time until break down (in years) of a particular Ipad, then X has the p.d.f

$$f_X(x) = \frac{1}{5}e^{-\frac{1}{5}x}, 0 \leq x < \infty. \tag{1}$$

Suppose an Apple store has 30 Ipads, and the distribution of their break down times are i.i.d. What, then, is the approximate probability that the average break down time of these Ipads is before 2 years?

Ans: One can check that $E(X) = 5$ and $Var(X) = 25$. Therefore $\bar{X}$ has distribution $N(5, \frac{25}{30})$. So

$$P(\bar{X} \leq 2) = P(Z \leq \frac{2-5}{\sqrt{\frac{25}{30}}}) \approx P(Z \leq -3.28) \approx 0.$$

(b) (10 points) There are 120 students in a Calculus 1 class at Rutgers. Suppose that each student has a probability of .3 of getting an A in this class, and the students' performance is independent of one another. What, then, is the approximate probability that the class will have at least 20 students getting an A ?

Ans: Let $X = Bin(.3, 120)$. Then $E(X) = 36$ and $Var(X) = 25.2$. by the normal approximation to the Binomial we have

$$P(X \geq 20) \approx P(Z \geq \frac{20-36}{\sqrt{25.2}}) \approx P(Z \geq -3.18) \approx 1.$$

In Questions 3,4,5 and 6, consider the multiperiod Binomial model where $u = \frac{4}{3}, d = \frac{2}{3}, r = 0, S_0 = 81$.

3. (15 points) Compute the price of the lookback option on S_k with expiration time $\mathbf{n = 3}$. Ans: $V_0 = 111.375$.

4. (15 points) Compute the price of the Asian option on S_k with expiration time $\mathbf{n = 3}$. Ans: $V_0 = 81$.

5. (15 points) Compute the price of the Down and Out option on S_k with expiration time $\mathbf{n = 3}$, the barrier $B = 70$, strike price $K = 120$.

   Ans: $V_0 = 9$.

6. (15 points) Compute the price of the American put option on S_k with expiration time $\mathbf{n = 3}$, strike price $K = 120$.

   Ans: $V_0 = 48$. In the following questions, let B_t be the standard Brownian motion, $B_0 = 0$. Compute:

7. (a) (5 points) $E(|B_t|)$.
   Ans:

$$\begin{aligned} E(|B_t|) &= \frac{2}{\sqrt{2\pi t}} \int_0^\infty x e^{-\frac{x^2}{2}} dx \\ &= \frac{2t}{\sqrt{2\pi t}} \Big(- e^{-u}\Big|_0^\infty \Big) (\text{ by substituting } u = \tfrac{x^2}{2t}) \\ &= \sqrt{\frac{2t}{\pi}}. \end{aligned}$$

   (b) (5 points) $E(e^{|B_t|})$

$$\begin{aligned} E(e^{|B_t|}) &= \frac{2}{\sqrt{2\pi t}} \int_0^\infty e^x e^{-\frac{x^2}{2}} dx \\ &= \frac{2e^{\frac{t}{2}}}{\sqrt{2\pi t}} \int_0^\infty e^{-\frac{(x-t)^2}{2t}} dx \\ &= 2e^{\frac{t}{2}} P(N(t,t) > 0) \\ &= 2e^{\frac{t}{2}} P(Z > -\frac{t}{\sqrt{t}}) \\ &= 2e^{\frac{t}{2}} N(\frac{t}{\sqrt{t}}). \end{aligned}$$

   (c) (5 points) $E(\cos(B_t))$.
   Since $e^{iB_t} = \cos(B_t) + i\sin(B_t)$,

$$E(e^{iB_t}) = E(\cos(B_t)) + iE(\sin(B_t)).$$

   But $E(e^{iB_t}) = e^{-\frac{t}{2}}$ and $E(\sin(B_t)) = 0$. So $E(\cos(B_t)) = e^{-\frac{t}{2}}$.

   (d) (5 points) $E(B_t^8)$.
   Ans: $105t^4$.

8. (a) (5 points) Let $S_t = t^2 e^{2B_t}$. Use Ito's formula to find dS_t^2.

Ans:

$$dS_t = 2te^{2B_t}dt + 2t^2e^{2B_t}dB_t + 2t^2e^{2B_t}dt.$$

So the volatility of $S_t, \sigma(t) = 2t^2e^{2B_t}$. Thus

$$\begin{aligned} dS_t^2 &= 2S_t dS_t + \sigma^2(t)dt \\ &= 2t^2e^{2B_t}\big((2te^{2B_t} + 2t^2e^{2B_t})dt + 2t^2e^{2B_t}dB_t\big) + 4t^2e^{4B_t}dt. \end{aligned}$$

(b) (5 points) Let $S_t = \cos(t^2 B_t)$. Use Ito's formula to find $d\cos(S_t)$.

Ans:

$$dS_t = -2tB_t \sin(t^2B_t)dt - t^2\sin(t^2B_t)dB_t - \frac{1}{2}t^4\cos(t^2B_t)dt.$$

So the volatility of $S_t, \sigma(t) = t^2\sin(t^2B_t)$. Thus

$$\begin{aligned} d\cos(S_t) &= -\sin(S_t)dS_t - \frac{1}{2}\cos(S_t)\sigma^2(t)dt \\ &= -\sin(\cos(t^2B_t))\big((-2tB_t\sin(t^2B_t) - \frac{1}{2}t^4\cos(t^2B_t))dt - t^2\sin(t^2B_t)dB_t\big) \\ &\quad -\frac{1}{2}\cos(S_t)t^4\sin^2(t^2B_t)dt. \end{aligned}$$

9. (a) (5 points) Let $r < s < t$. Find $E((B_t - B_s)(B_t + B_s)|B_r)$. Ans:

$$E((B_t - B_s)(B_t + B_s)|B_r) = E(B_t^2 - B_s^2|B_r).$$

We have showed in class $E(B_t^2|B_s) = B_s^2 + (t-s)$. So

$$E(B_t^2 - B_s^2|B_r) = B_r^2 + (t-r) - (B_r^2 + (s-r)) = t - s.$$

(b) (5 points) Let $s < t$. Find $E(e^{B_t^2}|B_s)$.
Ans: The solution for this problem is more complicated than I intended. So everyone will get full credit for this part.

10. Let S_t represent the stock price under the risk neutral measure using the Geometric Brownian motion model,i.e.

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

Let V_t represent the price at time 0 of the option that pays $V_T = \sqrt{S_T}$ at time T.

(a) (10 points) Compute $V_t = E(e^{-r(T-t)}\sqrt{S_T}|S_t)$ (The answer should be a function of t and S_t).
Ans: Recall that

$$S_T = S_t e^{(r-\frac{1}{2}\sigma^2)(T-t)+\sigma(B_T-B_t)}.$$

Thus

$$\begin{aligned} e^{-r(T-t)}\sqrt{S_T} &= \sqrt{S_t}e^{-r(T-t)+(\frac{1}{2}r-\frac{1}{4}\sigma^2)(T-t)+\frac{1}{2}\sigma(B_T-B_t)} \\ &= \sqrt{S_t}e^{(-\frac{1}{2}r-\frac{1}{4}\sigma^2)(T-t)+\frac{1}{2}\sigma(B_T-B_t)} \end{aligned}$$

So

$$\begin{aligned} V_t &= E(e^{-r(T-t)}\sqrt{S_T}|S_t) \\ &= \sqrt{S_t}e^{(-\frac{1}{2}r-\frac{1}{4}\sigma^2)(T-t)}e^{\frac{1}{8}\sigma^2(T-t)} \\ &= \sqrt{S_t}e^{(-\frac{1}{2}r-\frac{1}{8}\sigma^2)(T-t)}. \end{aligned}$$

(b) (10 points) Compute the Greeks $\Theta(t), \Delta(t), \Gamma(t)$ of this option.
Ans:

$$\begin{aligned} \Theta(t) &= \frac{\partial V_t}{\partial t} = (\frac{1}{2}r + \frac{1}{8}\sigma^2)\sqrt{S_t}e^{(-\frac{1}{2}r-\frac{1}{8}\sigma^2)(T-t)} \\ &= (\frac{1}{2}r + \frac{1}{8}\sigma^2)V_t \\ \Delta(t) &= \frac{\partial V_t}{\partial S_t} = \frac{1}{2\sqrt{S_t}}e^{(-\frac{1}{2}r-\frac{1}{8}\sigma^2)(T-t)} \\ &= \frac{V_t}{2S_t} \\ \Gamma(t) &= \frac{\partial^2 V_t}{\partial S_t^2} = \frac{-1}{4\sqrt{S_t^3}}e^{(-\frac{1}{2}r-\frac{1}{8}\sigma^2)(T-t)} = \frac{-V_t}{4S_t^2}. \end{aligned}$$

(c) (5 points) Show that V_t satisfies the Black-Scholes PDE. Ans: Black-Scholes PDE can be written as

$$\Theta(t) + \Delta(t) r S_t + \frac{1}{2}\Gamma(t)\sigma^2 S_t^2 - rV_t = 0.$$

Plug in the answers in part b:

$$\begin{aligned} &= (\frac{1}{2}r + \frac{1}{8}\sigma^2)V_t + \frac{V_t}{2S_t} r S_t + \frac{1}{2}\frac{-V_t}{4S_t^2}\sigma^2 S_t^2 - rV_t \\ &= V_t\big(\frac{1}{2}r + \frac{1}{8}\sigma^2 + \frac{r}{2} - \frac{1}{8}\sigma^2 - r\big) = 0. \end{aligned}$$

So the PDE is verified.

11. (5 points) Verify that the following Ito's integral is a martingale using Ito's formula: $\int_0^t B_s^2 dB_s$. Ans: See answer in homework 7.

12. (10 points) Let S_t be a geometric Brownian motion:

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

Find an explicit formula for the price at time 0 of a European put option on S_T with strike price K (Recall the put-call parity says price of Euro call + price of Euro put = price of the forward contract).

Ans: Since $(S_T - K)^+ - (S_T - K)^- = S_T - K$. We have

$$V_0^{call} - V_0^{put} = S_0 - Ke^{-rT}.$$

So

$$\begin{aligned} V_0^{put} &= V_0^{call} - S_0 + Ke^{-rT} \\ &= S_0(N(d_1) - 1) - Ke^{-rT}(N(d_2) - 1) \\ &= Ke^{-rT}N(-d_2) - S_0 N(-d_1) \end{aligned}$$

13. Let S_t be a geometric Brownian motion:

$$\begin{aligned} dS_t &= rS_t dt + \sigma S_t dB_t \\ S_0 &= 1000. \end{aligned}$$

Suppose $r = .1, \sigma = 0.25, T = 1, K = 1200$.

Follow the procedure described in class, compute:

(a) (10 points) The price of the Euro call on S_T with strike price K using the Black-Scholes formula.

$$d_1 = -.2043, d_2 = -.4543, V_0 = 66.3.$$

(b) (10 points) The price of the Euro call on S_T with strike price K using the Binomial approximation to the Black-Scholes model with 5 steps.

Since we use 5 steps, =0.2. So $S_{k+1} = S_k X_k$ where

$$\begin{aligned} X_k &= 1 + r\delta + \sigma\sqrt{\delta} = 1.1318 \text{ with probability } \frac{1}{2} \\ X_k &= 1 + r\delta - \sigma\sqrt{\delta} = 0.9082 \text{ with probability } \frac{1}{2} \end{aligned}$$

The terminal values of the stocks that give non-zero option values are: 1857 (5 ups), 1490 (4 ups 1 down). Thus the binomial approximation price is

$$e^{-.1}(657 + 5 \times 290) \times \frac{1}{2^5} = 59.5.$$