Math 485

Financial modeling in discrete time

A. Framework for modeling uncertain markets

Consider a market in M assets A model will do two things, at least:

1. It will specify all possible future histories, that is, outcomes, of the market. Notation:

$$\Omega = \text{the set of market histories.}$$

2. For each asset i, future market outcome ω, and future time t, it will define a price $S_t^{(i)}(\omega)$ for a unit of asset i.

Example: One period, one asset, binomial model. Despite its simplicity, even naiveté, the following model is basic to the course!

• The time periods of the model are $t = 0$ (today, the beginning of the period) and $t = 1$, some unit of time later (the end of the period).

• In the first period, there are two possible market outcomes only, a market upswing, which we denote u, or a market downswing, denoted d.

• If an upswing occurs, the asset return is g.
If a downswing occurs, the asset return is $\ell < g$.

Mathematically this translates to:

$$\begin{aligned} \Omega &\triangleq \{u, d\} \\ S_0 &\triangleq \text{ today's price, read from market.} \\ S_1(u) &\triangleq gS_0; \\ S_1(d) &\triangleq \ell S_0. \end{aligned}$$

Example: Extension to two periods:

• Periods $t = 0$, $t = 1$, $t = 2$.
• In each period, an upswing or downswing from previous market state.
• In each period, upswing implies return g, downswing return ℓ.

Model:

$$
\begin{aligned}
\Omega &\triangleq \{(u,u),(u,d),(d,u),(d,d)\} \\
S_0 &\triangleq \text{ today's price;} \\
S_1(u,u) &= S_1(u,d) \triangleq gS_0 \\
S_1(d,u) &= S_1(d,d) \triangleq \ell S_0; \\
S_2(u,u) &\triangleq g^2 S_0 \\
S_2(u,d) &= S_2(d,u) \triangleq g\ell S_0 \\
S_2(d,d) &\triangleq \ell^2 S_0
\end{aligned}
$$

B. Discrete probability spaces

Discrete probability spaces are a framework for modeling an experiment with a random outcome, when the number of possible outcomes is finite. To define a discrete probability space:

1. Define the *outcome space* to be the set of all possible outcomes.

• Notation: $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$.

• Terminology: Subsets of Ω are called *events*. To say, of a trial, "event A occurs" means "the trial's outcome belongs to subset A."

2. To each ω_i in Ω, assign a number p_i, representing the probability that ω_i is the outcome: Require:

(a) for each i, $0 \le p_i \le 1$;

(b) $\sum_{1}^{N} p_i = 1$.

For each event A, define

$$\mathbb{P}(A) \triangleq \sum_{\omega_i \in \Omega} p_i.$$

More terminology:

- $\mathbb{P}$ is called a probability measure on the set of events.
- Ω and $\mathbb{P}$ together constitute a *probability space*.

Remark: $\mathbb{P}(\{\omega_i\}) = p_i$. We often write this as $\mathbb{P}(\omega_i)$.

The entire construction goes through in exactly the same way if Ω is countably infinite: $\Omega = \{\omega_1, \omega_2, \ldots\}$. Requirement (b) becomes

$$\sum_1^\infty p_i = 1.$$

Probability spaces in general. The probability measures, as just defined on finite or countably infinite Ω, satisfy the **finite additivity property:** if $A_1, A_2, \ldots, A_k$ are *disjoint* events,

$$I\!\!P\left(A_1 \cup \cdots \cup A_k\right) = \sum_{i=1}^k I\!\!P(A_i). \tag{1}$$

If Ω is countably infinite, $I\!\!P$ also is *countably additive*: if $A_1, A_2, \ldots,$ are *disjoint* events,

$$I\!\!P\left(A_1 \cup A_2 \cup \cdots\right) = \sum_{i=1}^\infty I\!\!P(A_i). \tag{2}$$

In the general definition of a probability space, identities (1) and (2) are taken as *axioms*.

Example. Adding probabilities to the two period, binomial market model.

Recall that in this case $\Omega = \{(u,u), (u,d), (d,u), (d,d)\}$

Model I: After research and observation we think
$I\!\!P\left((u,u)\right) = \frac{1}{2}, \quad I\!\!P\left((u,d)\right) = \frac{1}{4}, \quad I\!\!P\left((d,u)\right) = \frac{1}{8}, \quad I\!\!P\left((d,d)\right) = \frac{1}{8}.$

Problem. Let A be the event of upswing in the first period. Find $I\!\!P(A)$.

Note that $A = \{(u,u), (u,d)\}$. Thus $I\!\!P(A) = I\!\!P((u,u)) + I\!\!P((u,d)) = 1/2 + 1/4 = 3/4$.

Model II: (Random Walk, Bull Market) Assume the probability of an upswing in each period is $3/4$ *and* market movements in different periods are *independent*. Then
$I\!\!P\left((u,u)\right) = (\frac{3}{4})^2, \quad I\!\!P\left((u,d)\right) = \frac{3}{4}\left(\frac{1}{4}\right),$
$I\!\!P\left((d,u)\right) = \frac{1}{4}\left(\frac{3}{4}\right), \quad I\!\!P\left((d,d)\right) = (\frac{1}{4})^2.$

Problem. Find $I\!\!P$ (at least one upswing).

If B is the event of at least one upswing in the two periods, the complement B^c of B is the event of two downswings, which is the singleton event $\{(d,d)\}$. Thus $I\!\!P(B) = 1 - I\!\!P(B^c) = 1 - (1/8) = 7/8$.

C. Discrete Random Variables

Object: Model an experiment whose random outcome is a real number in the set $\mathcal{E} = \{y_1, \ldots, y_M\}$.

Approach: Label the outcome of a hypothetical trial by X. X is an example of a **random variable**. The complete description of the behavior of X is given by its **probability mass function**

$$p_X(y), \qquad y \in \mathcal{E},$$

where for each y, $p_X(y)$ gives the probability that X equals y. We write also $I\!\!P(X = y)$.

Of course, we require $\sum_{y \in \mathcal{E}} p_X(y) = 1$.

For any subset U of real numbers, we define

$$I\!\!P(X \in U) \stackrel{\triangle}{=} \sum_{y \in U} p_X(y).$$

Expectation

The *expected value* or *mean* of X is

$$E[X] \stackrel{\triangle}{=} \sum_{y \in \mathcal{E}} y p_X(y).$$

The *law of the unconscious statistician* says that for any function g:

$$E[g(X)] = \sum_{y \in \mathcal{E}} g(y) p_X(y).$$

Example: X is Bernoulli(p) if $I\!\!P(X = 1) = p$, $\underline{(}X = 1) = p$. Then

$$\mu = E[X] = 0 \cdot (1-p) + 1 \cdot p = b.$$

$$\text{Var}(X) \stackrel{\triangle}{=} E\left[(X-\mu)^2\right] = E[X^2] - p^2 = p(1-p).$$

Functions on probability spaces give r.v.'s

In this course, random variables will often arise as functions defined on a probability space. Here is an example.

Example: This is the random walk, bull market model continued, but now we add prices of a risky asset according to:

$$
\begin{aligned}
\Omega &\triangleq \{(u,u),(u,d),(d,u),(d,d)\} \\
S_1(u,u) &= S_1(u,d) \triangleq gS_0 \\
S_1(d,u) &= S_1(d,d) \triangleq \ell S_0; \\
S_2(u,u) &\triangleq g^2 S_0 \\
S_2(u,d) &= S_2(d,u) \triangleq g\ell S_0 \\
S_2(d,d) &\triangleq \ell^2 S_0 \\
I\!P\left((u,u)\right) &= (\frac{3}{4})^2, \quad I\!P\left((u,d)\right) = \frac{3}{4}\left(\frac{1}{4}\right) \\
I\!P\left((d,u)\right) &= \frac{1}{4}\left(\frac{3}{4}\right), \quad I\!P\left((d,d)\right) = (\frac{1}{4})^2.
\end{aligned}
$$

S_1, the price at time 1, and S_2 are random variables! We can compute their probability mass functions from the probability measure $I\!P$.

For example, suppose $S_0 = 20$, $g = 1.05$, $\ell = .95$. Then $S_1((u,u)) = S_1(u,d) = 20(1.05) = 21$ and $S_1((d,u)) = S_1((d,d)) = 19$.
The probability mass function of S_1 is

$$p_1(21) = \frac{3}{4} \qquad p_1(19) = \frac{1}{4}.$$

Its expectation is $E[S_1] = 21(3/4) + 19(1/4) = 20.5$.

For S_2: $S_2((u,u)) = 20(1.05)^2 = 22.05$, $S_2((u,d)) = S_2((d,u)) = 20(.95)(1.05) = 19.95$, and $S_2((d,d)) = 20(.95)^2 = 18.05$.
The probability mass function and expectation of S_2 are:

$$p_2(18.05) = \frac{1}{16},\ p_2(19.95) = \frac{6}{16},\ p_3(22.05) = \frac{9}{16}.$$

$$E[S_2] = \frac{18.05}{16} + \frac{6(19.95)}{16} + \frac{9(22.05)}{16} = 21.0125.$$