GUIDE TO WRITING MATHEMATICAL PROOFS

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1. Introduction

There is no general prescribed format for writing a mathematical proof. Some methods of proof, such as Mathematical Induction, involve the same steps, though the steps themselves may require their own methods of proof.

A mathematical statement may also have several proofs using different methods. One method is generally preferred over another if it is shorter, simpler or clearer.

Proofs are often arrived at by trial and error, writing and revision. They may involve a 'creative step' or 'new idea'.

Here we discuss some general rules for writing proofs and an overview of techniques of proof.

1.1. **General rules for writing proofs.**

All written proofs should begin by establishing notation and recalling assumptions. We may also recall a definition if it is used within a proof.

A useful rule to keep in mind is that each new mathematical claim that is not well known should be justified by reasoning or a reputable reference.

Here are some general rules to remember when writing proofs.

- The audience for a written proof should be considered to be a group of one's peers.
- It is customary to declare your method of proof in order to inform the reader. 'For the sake of contradiction, we assume that. . . '
- Always define a new symbol: 'Let x denote a natural number...' .
- Never reuse the same letter for a different meaning in the same proof.
- We may reuse a letter after a proof is complete, but it should always be redefined in its new context.
- A proof must read as a logical sequence of steps.
- It must make sense in English when read out aloud.
- A proof should read as a balance of words and symbols.
- A proof must be grammatically correct in English.
- Do not begin a sentence with a mathematical symbol.
- The status of each mathematical statement should be explicitly declared: 'We assume that. . . ', 'We must show that. . . ', 'We conclude that. . . ', 'From our previous argument. . . '
- Quantifiers should be used to indicate the truth of a statement: 'For all $x \in \mathbb{Z}$ we have. . . ', 'There exists $x \in \mathbb{Z}$ satisfying. . . '
- Any method of proof may use earlier theorems. They should be cited by a reputable source. For example you may refer to the lecture notes or other class material.
- Inform the reader when the proof is complete: 'This completes the proof.' Or use QED or $\square$

2. Methods of proof

We can't know in advance what method of proof is the most suitable to use for any given mathematical statement. A successful proof is often the result of several attempts, possibly having tried different methods.

2.1. **Direct proof.**

A *direct proof* is a proof that establishes the result using direct arguments and deductions that are natural consequences of the definitions and assumptions.

If the statement under consideration is of the form 'If P then Q' or '$P \implies Q$', a direct method of proof begins by assuming that P is true. This is followed by a sequence of statements and deductions using definitions, assumptions and logical equivalences that lead to a conclusion that Q is true.

2.2. **The contrapositive form of a direct implication.**

The direct implication $P \implies Q$ is logically equivalent to its *contrapositive form*. This is the statement

$$\neg Q \implies \neg P$$

where $\neg P$ is the *negation* of the statement P. To *prove* a direct implication $P \implies Q$, we may prove instead the contrapositive form $\neg Q \implies \neg P$.

In some cases, a proof of the contrapositive form of an implication is easier to obtain than a proof of the direct implication itself. This is often the case if the conclusion Q is a compound statement using the connectives $\wedge$ or $\vee$.

That is, if we wish to prove propositions of the form

$$P \implies Q \wedge R$$

or

$$P \implies Q \vee R$$

it may be more convenient to prove the contrapositive forms

$$\neg Q \vee \neg R \implies \neg P$$

or

$$\neg Q \wedge \neg R \implies \neg P$$

respectively.

The term *contraposition* is often used for the method of proof which involves proving the contrapositive form of a direct implication.

2.3. **Proof by contradiction.**

To prove a conclusion Q under assumptions $\mathcal{A}$, we first suppose that $\mathcal{A}$ is true but Q is false. We follow the natural deductions from these assumptions and reach an inconsistency called a *contradiction*, often in the form of a statement and its opposite both holding. We deduce that under assumptions $\mathcal{A}$, the conclusion Q must be true.

In particular, to prove that $P \implies Q$ is true using proof by contradiction, we proceed by assuming that $P \implies Q$ is false and obtaining a contradiction. This is equivalent to assuming P and the negation of Q, that is, assuming that P is true and Q is false and obtaining a contradiction.

2.4. **'Sketch proofs'.**

A 'sketch proof' is an intuitive outline of a proof to give an overview of the main ideas when the full details are beyond scope.

2.5. **Proving a bi-conditional statement.** Recall that the symbol $\iff$ denotes mathematical equivalence. That is, if P and Q are mathematical statements, then $P \iff Q$ is defined as

$$(P \implies Q) \text{ and } (Q \implies P).$$

That is, to prove a statement of the form $P \iff Q$, or P if and only if Q, it is necessary to prove the implications $P \implies Q$ and $Q \implies P$. A statement of the form $P \iff Q$ is called a 'bi-conditional statement'.

It is not enough to prove only one of the implications $P \implies Q$ or $Q \implies P$. This is because the truth of one of these implications is not related to the truth of the other. They must both be verified independently.

Within a proof of a bi-conditional statement, you may use any method to prove the implications $P \implies Q$ and $Q \implies P$.

2.6. **Disproving by counterexamples.** A *counterexample* is an exception to a proposed general rule.

The rule may be true in many instances, but if there is a single example where it fails, we say that the rule is *false in general*. We may thus disprove a mathematical statement by demonstrating a single counterexample.

2.7. **Correcting a statement that has counterexamples.** A statement that has counterexamples can often be repaired so as to eliminate the counterexamples. For example, we may restrict the universe of values for which the statement holds, or restrict the conclusion to give a true statement in the given universe.

2.8. **Proof by cases.** Sometimes a proof can be carried out by breaking the possibilities up into several cases and writing a separate proof for each case. Natural choices of cases for statements involving $n \in \mathbb{Z}$ could be $n \in \mathbb{E}$ and $n \in \mathbb{O}$, or $n < 0$ and $n \geqslant 0$.

2.9. **Constructive proof.** A *constructive proof* demonstrates the existence of a mathematical object by constructing it explicitly and showing that it has the required properties. More explicitly, this is a proof of a statement of the form $(\exists x \in A)(P(x))$. Such a proof involves constructing an element x in a set A and showing that it satisfies property $P(x)$.

2.10. **Existential proof.** An *existential proof* establishes the existence of a mathematical object satisfying certain properties without constructing it explicitly. More explicitly, this is a proof of a statement of the form $(\exists x \in A)(P(x))$. Such a proof argues the existence of such an element x in a set A satisfying property $P(x)$ but does not show how to construct x.