Math 300 Intro Math Reasoning
Worksheet 9: Equinumerability

(1)

(1) Let $Y = \{n+2 \mid n \in \mathbb{N}\} \subseteq \mathbb{N}$. Find a bijective map $f : \mathbb{N} \longrightarrow Y$.
(2) Show that $(-1, 2) \sim (6, 7)$
(3) Find an injection from $\mathbb{N} \times \mathbb{N}$ into $P(\mathbb{N})$.

Solution.

(1) $f(n) = n + 2$.
(2) $f(x) = \frac{x+1}{3} + 6$.
(3) $f : \mathbb{N} \times \mathbb{N} \to P(\mathbb{N})$ $f(\langle n, m\rangle) = \{n, n+m\}$.

(2) Suppose that $A \sim A'$ and $B \sim B'$. Prove that $A \times B \sim A' \times B'$

Solution. Let $f : A \to B$ and $g : C \to D$ be bijections. Define $h : A \times C \to B \times D$ $h(\langle a, c\rangle) = \langle f(a), g(c)\rangle$. Prove that h is one-to-one. Let us prove for example that h is onto. Let $\langle b, d\rangle \in B \times D$. Since f, g are onto, there are $a \in A$ and $c \in C$ such that $f(a) = b$ and $g(c) = d$. Then $\langle a, c\rangle \in A \times C$ and $h(\langle a, c\rangle) = \langle f(a), g(c)\rangle = \langle b, d\rangle$.

(3) Prove that $P(\mathbb{N} \times \mathbb{Z}) \sim {}^{\mathbb{N}}\{0, 1\}$.

Solution By a theorem from class $\mathbb{Z} \sim \mathbb{N}$ and therefore by a theorem from class (which is also the next problem) if $A \sim A'$ and $B \sim B'$ then $A \times B \sim A' \times B'$. In particular $\mathbb{N} \times \mathbb{N} \sim \mathbb{N} \times \mathbb{Z}$. Another theorem from class is used to show $\mathbb{N} \times \mathbb{N} \sim \mathbb{N}$ and another one, to show that if $A \sim B$ then $P(A) \sim P(B)$. So we conclude that since $\mathbb{N} \times \mathbb{Z} \sim \mathbb{N}$, $P(\mathbb{N} \times \mathbb{Z}) \sim P(\mathbb{N})$. Finally, we prove that for every set A, ${}^{A}\{0, 1\} \sim P(A)$ and therefore $P(\mathbb{N} \times \mathbb{Z}) \sim P(\mathbb{N}) \sim {}^{\mathbb{N}}\{0, 1\}$.

(4) Suppose that A is countable (and infinite) and $a \notin A$. Show that $A \cup \{a\} \sim A$

Solution.

Let $f : \mathbb{N} \to A$ be a bijection which exists since A is countable. To see that $A \cup \{a\} \sim A$, it is enough to see that $A \cup \{a\} \sim \mathbb{N}$. Deinfe $g : A \cup \{a\} \to \mathbb{N}$ by $g(x) = \begin{cases} f(x) + 1 & x \neq a \\ 0 & x = a \end{cases}$
Let us prove that g is a bijection. To see that g is $1-1$, let $x_1, x_1 \in A \cup \{a\}$ and assume that $g(x_1) = g(x_2)$. If $g(x_1) = g(x_2) > 0$, then $x_1, x_2 \neq 0$ and therefore $f(x_1) + 1 = g(x_1) = g(x_2) = f(x_2) + 1$ hence $f(x_1) = f(x_2)$ and since f is injective $x_1 = x_2$. If $g(x_1) = g(x_2) = 0$, then it must be that $x_1 = a = x_2$ by the definition of g.

To see that g is onto, let $n \in \mathbb{N}$. If $n = 0$, then $g(a) = 0$. If $n > 0$, let $x \in A$ be such that $f(x) = n - 1$ which exists by the assumption that f is onto. Then $g(x) = f(x) + 1 = n - 1 + 1 = n$ as wanted.

(5) Suppose that $A \sim A'$, $B \sim B'$. Show that ${}^{A}B \sim {}^{A'}B'$.

solution

Let $f : A \to A'$ and $g : B \to B'$ be bijections and let $f^{-1} : A' \to A$, $g^{-1} : B' \to B$ be the inverse functions which exist by the theorem from class. Define $h : {}^{A}B \to {}^{A'}B'$ by $f(\phi) = g \circ \phi \circ f^{-1}$. To show that h is a bijection let us find an inverse to h. Define $k : {}^{A'}B' \to {}^{A}B$ by $k(\psi) = g^{-1} \circ \psi \circ f$.

So $k(h(\phi)) = g^{-1} \circ g \circ \phi f^{-1} \circ f = \phi$ and the other direction is proven similarily.