Math 300 Intro Math Reasoning
Worksheet 08: Set Theory

(1) Let $F : {}^{\mathbb{N}}\mathbb{N} \to \mathbb{N}$ be the function $F(f) = f(0)$. Is F 1-1? is F onto?

Solution: F is not $1-1$ since for example the function $f(n) = 0$ and $id_{\mathbb{N}}$ are different function as for example $f(1) = 0 \neq 1 = id_{\mathbb{N}}(1)$ but $F(f) = f(0) = 0 = id_{\mathbb{N}}(0) = F(id_{\mathbb{N}})$.

F is onto: Let $n \in \mathbb{N}$ WTP there is $f \in {}^{\mathbb{N}}\mathbb{N}$ such that $F(f) = n$. Let f be the constant function $f(x) = n$. Then $F(f) = f(0) = n$.

(2) Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = \begin{cases} 5x & x < 0 \\ x^2 & x \geq 0 \end{cases}$

(1) What is $f \circ f$?
(2) Show that f is invertible.
(3) Compute f^{-1}

Solution: For (1) we have that $f \circ f(x) = \begin{cases} 25x & x < 0 \\ x^4 & x \geq 0 \end{cases}$.

For 2, to see that f is invertible let us prove it is one to one and onto. To see it is one to one, let $x_1, x_2 \in \mathbb{R}$. Suppose that $f(x_1) = f(x_2)$ WTP $x_1 = x_2$. Let us split into cases:

(1) If $f(x_1) = f(x_2) \geq 0$, then $x_1, x_2 \geq 0$ and therefore $x_1^2 = f(x_1) = f(x_2) = x_2^2$. Hence $x_1 = \pm x_2$. Since x_1, x_2 are positive, it follows that $x_1 = x_2$.
(2) If $f(x_1) = f(x_2) < 0$, then $x_1, x_2 < 0$ and therefore $5x_1 = f(x_1) = f(x_2) = 5x_2$, hence $x_1 = x_2$.

It follows that f is $1-1$. To see that f is onto, let $y \in \mathbb{R}$ WTP there is $x \in \mathbb{R}$ such that $f(x) = y$. If $y \geq 0$, let $x = \sqrt{y}$, then $x \geq 0$ and therefore $f(x) = x^2 = y$. If $y < 0$, let $x = \frac{y}{5}$, then $x < 0$ and $f(x) = 5x = y$ as wanted.

Since f is one to one and onto, by a theorem from class it is invertible.

Finally, we compute $f^{-1}(y) = \begin{cases} \sqrt{y} & y \geq 0 \\ \frac{y}{5} & y < 0 \end{cases}$.

(3) Find bijections:

(1) $\mathbb{N}$, $(\mathbb{N} \setminus \{0, 1\}) \cup \{-5\}$.
(2) $\mathbb{R}$, $(-\infty, 0] \cup (1, \infty)$.

Solution: For (1), define $f : \mathbb{N} \to \mathbb{N} \setminus \{0, 1\} \cup \{-5\}$ by $f(n) = \begin{cases} -5 & n = 0 \\ n + 1 & n \geq 1 \end{cases}$.

for 2, define $f : \mathbb{R} \to (-\infty, 0] \cup (1, \infty)$ by $f(x) = \begin{cases} x & x \leq 0 \\ x + 1 & x > 0 \end{cases}$.