Math 300 Intro Math Reasoning
Worksheet 05: Set Theory

(1) Prove by induction that

$$1 \cdot 2 + 3 \cdot 4 + ... + (2n-1) \cdot 2n = \frac{n(n+1)(4n-1)}{3}$$

Solution: By induction.

Base $n = 1$ $1 \cdot 2 = \frac{1 \cdot 2 \cdot 3}{3}$.
I.H. Assume that

$$1 \cdot 2 + 3 \cdot 4 + ... + (2n-1) \cdot 2n = \frac{n(n+1)(4n-1)}{3}$$

Step Let us prove that

$$1 \cdot 2 + 3 \cdot 4 + ... + (2n-1) \cdot 2n + (2n+1)(2n+2) = \frac{(n+1)(n+2)(4n+3)}{3}$$

Indeed,

$$1 \cdot 2 + 3 \cdot 4 + ... + (2n-1) \cdot 2n + (2n+1)(2n+2) = \frac{n(n+1)(4n-1)}{3} + 2(2n+1)(n+1) =$$

$$\frac{(n+1)[n(4n-1) + 6(2n+1)]}{3} = \frac{(n+1)(4n^2 + 11n + 6)}{3} = \frac{(n+1)(n+2)(4n+3)}{3}$$

as wanted.

(2) Prove or disprove:

(1) $P(A) \cap P(B) = P(A \cap B)$.
(2) $P(A) \cup P(B) = P(A \cup B)$

Solution: For (1) we prove that $P(A \cap B) = P(A) \cap P(B)$ by a double inclusion.

$\underline{P(A \cap B) \subseteq P(A) \cap P(B)}$: Let $X \in P(A \cap B)$ WTP $X \in P(A) \cap P(B)$. By definition of powerset, $X \subseteq A \cap B$. Since $A \cap B \subseteq A$ and $A \cap B \subseteq B$, it follows that $X \subseteq A$ and $X \subseteq B$. Hence $X \in P(A)$ and $X \in P(B)$ and by definition of intersection $X \in P(A) \cap P(B)$.

$\underline{P(A) \cap P(B) \subseteq P(A \cap B)}$: Let $X \in P(A) \cap P(B)$, WTP $X \in P(A \cap B)$. By assumption $X \in P(A)$ and $X \in P(B)$, and therefore $X \subseteq A$ and $X \subseteq B$. To see that $X \subseteq A \cap B$, let $x \in X$, the $x \in A$ and $x \in B$ (since $X \subseteq A$ and $X \subseteq B$) and therefore $x \in A \cap B$. It follows that $X \subseteq A \cap B$, namely $X \in P(A \cap B)$.

For (2) we disprove, take $A = \{1\}$ and $B = \{2\}$. Let us explain although it was not requires in the exam. Then $\{1,2\} \in P(\{1,2\}) = P(A \cup B)$ and $\{1,2\} \notin P(\{1\}) = P(A)$

and $\{1,2\} \notin P(\{2\}) = P(B)$, hence $\{1,2\} \notin P(A) \cup P(B)$. It follows that $P(A \cup B) \neq P(A) \cup P(B)$.

(2) Prove by induction that for every n, $2^{n+1} + 5^n$ is divisible by 3.

Solution: For the base case, $2^{0+1} + 5^0 = 2 + 1 = 3$ is divisible by 3.

The induction hypothesis: Suppose that $2^{n+1} + 5^n$ is divisible by 3

Induction step, let us prove that $2^{n+2} + 5^{n+1}$ is divisible by 3. By the induction hypothesis there is l such that $2^{n+1} + 5^n = 3l$. Now for $n+1$ we have,

$$2^{n+2} + 5^{n+1} = 2(2^{n+1}) + 5(5^n) = 2(2^{n+1} + 5^n) + 3(5^n) = 2(3l) + 3(5^n) = 3(2l + 5^n).$$

Define $k = 2l + 5^n$, then k is an integer. Since we have shown that $2^{n+2} + 5^{n+1} = 3k$, $2^{n+2} + 5^{n+1}$ is divisible by 3