Math 300 Intro Math Reasoning
Worksheet 05: Set Theory

(1) Prove that for every three sets A, B, C, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

Solution. This is set equality. We prove it by a chain of equivalences.

Let x be any element.

$$x \in A \cup (B \cap C) \underset{\text{def of } \cup}{\iff} (x \in A) \vee (x \in B \cap C) \underset{\text{def of } \cap}{\iff} (x \in A) \vee ((x \in B) \wedge (x \in C))$$

$$\underset{\text{logical identity } \alpha\vee(\beta\wedge\gamma)\equiv(\alpha\vee\beta)\wedge(\alpha\vee\gamma)}{\iff} ((x \in A) \vee (x \in B)) \wedge ((x \in A) \vee (x \in C))$$

$$\underset{\text{def of } \cup}{\iff} (x \in A \cup B) \wedge (x \in A \cup C) \underset{\text{def of } \cap}{\iff} x \in (A \cup B) \cap (A \cup C)$$

(2) Prove that $\mathbb{Z} = \{r \in \mathbb{R} \mid \mathbb{Z} \cap (r, r+1) = \emptyset\}$

[Hint: you may use the fact that for any $z \in \mathbb{Z}$ there are no integers between z and $z+1$. You may also use “round up” and “round down”.]

Solution. We prove this set equality by double inclusion. Denote the set of the right hand side by A.

(1) $\underline{\mathbb{Z} \subseteq A}$: Let $z \in \mathbb{Z}$ WTP $z \in \mathbb{R}$. By separation (definition of A) WTP $z \in \mathbb{R} \wedge \mathbb{Z} \cap (z, z+1) = \emptyset$. Clearly $z \in \mathbb{R}$. Suppose towards a contradiction that $\mathbb{Z} \cap (z, z+1) \neq \emptyset$ and let $z' \in \mathbb{Z} \cap (z, z+1)$, then z' is an integer strictly between an integer z and its successor, contradicting the ”hint”. Hence $z \in A$.
(2) $A \subseteq \mathbb{Z}$*:* Let $r \in A$ WTP $r \in \mathbb{Z}$. By separation it follows that $r \in \mathbb{R}$ and that $\mathbb{Z} \cap (r, r+1) = \emptyset$. Suppose towards contradiction that $r \notin \mathbb{Z}$, then $r < \lceil r \rceil < r+1$ and by definition $\lceil r \rceil \in \mathbb{Z} \cap (r, r+1)$. Hence $\mathbb{Z} \cap (r, r+1) \neq \emptyset$, contradiction. Hence $r \in \mathbb{Z}$.

We define for every $A \subseteq \mathbb{R}$ and $r \in \mathbb{R}$

$$A + r = \{a + r \mid a \in A\}$$

(3) Compute (not proof):

(1) $\{1, 5\} + 0.5 = \{1.5, 5.5\}$.
(2) $\mathbb{N} + 1 = \mathbb{N}_+$.
(3) $\mathbb{Z} + 1 = \mathbb{Z}$.
(4) $\emptyset + r = \emptyset$.

(4) Prove or disprove:

(1) If $A \subseteq B$ then $A + r \subseteq B + r$.

Solution. Suppose that $A \subseteq B$. WTP $A + r \subseteq B + r$. Let $x \in A + r$. WTP $x \in B + r$. By the replacement principle, there is $a \in A$ such that $x = a + r$. Since $A \subseteq B$, $a \in B$. Therefore there is $b \in B$ such that $x = b + r$ which again by replacement implies that $x \in B + r$.

(2) If for some $r, s \in \mathbb{R}$, $A + r \subseteq B + s$ then $A \subseteq B$.

Solution. This is false, for every $A = \{1\}$, $B = \{2\}$ then $\{1\} + 1 \subseteq \{2\} + 0$, however $\{1\} \not\subseteq \{2\}$.

(3) $A + 0 = A$

Solution. Let us prove this set equality by a double inclusion.

- $\supseteq$: Let $a \in A$. WTP $a \in A + 0$. We have that $a = a + 0$, and therefore there is $x \in A$ such that $x + 0 = a$. By replacement, $a \in A + 0$.
- $\subseteq$ For the other direction, suppose that $x \in A + 0$, the by replacement, there is $a \in A$ such that $x = a + 0 = a$ and therefore $X \in A$.

Since we proved a double inclusion it follows that $A + 0 = A$

(5) Prove that for every $r \in \mathbb{R}$, $\mathbb{Q} + r = \mathbb{Q}$ if and only if $r \in \mathbb{Q}$.

Solution. Let $r \in \mathbb{R}$. Let us prove the equivalence by a double implication:

- $\Rightarrow$ Suppose that $\mathbb{Q}+r = \mathbb{Q}$. WTR $r \in \mathbb{Q}$. Note that by replacement, $r = 0+r \in \mathbb{Q}+r$. By the set equality assumption $r \in \mathbb{Q}$.
- $\Leftarrow$ Suppose that $r \in \mathbb{Q}$ and let us prove that $\mathbb{Q} + r = \mathbb{Q}$ by a double inclusion.
  - $\subseteq$ Let $x \in \mathbb{Q} + r$. WTP $x \in \mathbb{Q}$. By replacement, there is $q \in \mathbb{Q}$ such that $x = q + r$. Since both q and r are rationals, $x \in \mathbb{Q}$.
  - $\supseteq$ Let $x \in \mathbb{Q}$ WTP $x \in \mathbb{Q}+r$. Define $q = x-r$, the again since x, r are rationals, $q \in \mathbb{Q}$. Also note that $x = q + r$ and therefore $x \in \mathbb{Q} + r$.

  Since we proved a double inclusion it follows that $\mathbb{Q} = \mathbb{Q} + r$,