Math 300 Intro Math Reasoning
Worksheet 04: Mathematical logic

(1) Compute the negation and prove or disprove the following statement.(In $U = \mathbb{R}$)

$$\forall x(\forall y((x < y) \Rightarrow (\exists z(x < z \wedge z < y)))).$$

Solution.

$$\forall x(\forall y((x < y) \Rightarrow (\exists z(x < z \wedge z < y)))) \equiv$$
$$\equiv \exists x(\exists y(x < y \wedge \forall z(z \leq x \vee y \leq z)$$

Let us prove the statement, let $x, y \in \mathbb{R}$, suppose that $x < y$, WTP $\exists z(x < z < y)$. Define $z = \frac{x+y}{2}$. Then

$$x = \frac{x+x}{2} < \frac{x+y}{2} < \frac{y+y}{2} = y$$

.

(2) Prove that

$$\lim_{n\to\infty} \frac{n+1}{n} = 1$$

[Recall: $\lim_{n\to\infty} a_n = L$ means $\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n \geq N\ |a_n - L| < \epsilon$.]

Solution. Let $\epsilon > 0$. WTP $\exists N \in \mathbb{N} \forall n \geq N\ |\frac{n+1}{n} - 1| < \epsilon$. Define $N = \lceil 1/\epsilon \rceil + 1$. WTP $\forall n \geq N\ |\frac{n+1}{n} - 1| < \epsilon$. Let $n_0 \geq N = \lceil 1/\epsilon \rceil + 1$ Then

$$|\frac{n+1}{n} - 1| = \frac{1}{n} \leq \frac{1}{N} < \frac{1}{\frac{1}{\epsilon}} = \epsilon$$

(3) Prove that $1 + \sqrt{2}$ is irrational.

Solution. Suppose towards contradiction that $1 + \sqrt{2} = q \in \mathbb{Q}$. Then $\sqrt{2} = q - 1$ is rational as the difference of two rational numbers. This contradicts the theorem we saw in class that $\sqrt{2}$ is irrational. Hence $1 + \sqrt{2}$ is irrational. (3)

Prove that $\sqrt{3}$ is irrational.

proof. Assume otherwise, then there are coprime integers n, m (namely, n, n have no common factor) such that $\sqrt{3} = \frac{n}{m}$. Hence $3 = \frac{n^2}{m^2}$ and

$$(*)\ \ m^2 3 = n^2.$$

It follows that n^2 is divisible by 3. Let us prove that n must be divisible by 3. Otherwise, $n = 3k + i$ where k is an integer and $i = 1, 2$. Then $n^2 = 9k^2 + 6ki + i^2$. but then $n^2 - 9k^2 - 6ki = i^2$ and the number on the left hand side is divisible by 3 (as the difference of numbers which are divisible by 3). If $i = 1$, we get that $i^2 = 1$ is divisible by 3, a contradiction. If $i = 2$, then $i^2 = 4$ is divisible by 3, cotradiction. So we conclude that n is divisible by 3. Namely $n = 3r$ for some integer r. Substituting in equation $(*)$, we get

$$m^2 3 = 9r^2 \Rightarrow m^2 = 3r^2$$

By the same reasoning, m^2 is divisible by 3 and therefore m is divisible by 3. This is a contradiction of m, n being co prime. (4) $A = \{1, 2, 3\}$, $B = \{1, 1, 2, 3\}$,

$C = \{n \in \mathbb{N} \mid \exists y \in \mathbb{R}(|y| + |3 - n| \leq 3)\}$, $D = \{\{1\}, \{1, 2\}, \{1, 2, 3\}\}$, $E = \{1, \{1, 2, 3\}, 3\}$
$F = \{2^n - m \mid n \in \mathbb{N}, m \in \{0, 1\}\}$

(1) How many elements are in each of the sets?
(2) Determine if
   (a) $A = B$. True
   (b) $A \subseteq E$. False
   (c) $A \in E$. True
   (d) $A = C$. False
   (e) $A \subseteq C$ True
   (f) $E \subseteq D$. False
   (g) $A \subseteq F$ True
   (h) $C \subseteq F$? False