Math 300 Intro Math Reasoning
Worksheet 03: Mathematical logic

(1) Prove the following statement: An integer is divisible by 5 if and only if its last digit is divisible by 5.

[Hint: To formally refer to the unit number of an integer n, decompose $n = 10k + d$ where k is some integer and $0 \leq d \leq 9$. Then d is the unit digit of n.]

Solution Let n be an integer and suppose that $n = 10k + d$ where $k, d \in \mathbb{Z}$ and $0 \leq d \leq 9$. Let us prove the statement by a double implication.

- $\Rightarrow$: If 5 divides n, then $n = 5l$ for some $l \in \mathbb{Z}$ and therefore $d = n - 10k = 5l - 10k = 5(l - 2k)$. Since $l - 2k \in \mathbb{Z}$, 5 divides d.
- $\Leftarrow$: If 5 divides d, then $d = 5l$ for some $l \in \mathbb{Z}$. Then $n = 10k + 5l = 5(2k + l)$ and since $2k + l \in \mathbb{Z}$, 5 divides n.

(2) Prove that for all integers n and m, if n is multiple of 6 or m is multiple of 9 then n^2m is a multiple of 9.

Solution: Suppose that n is a multiple of 6 or m is a multiple of 9. WTP n^2m is a multiple of 9. Let us split into cases:

(1) If 6 divides n, then $n = 6k$ for some $k \in \mathbb{Z}$ and therefor $n^2m = (6k)^2m = 9(4k^2m)$. Since $4k^2m \in \mathbb{Z}$, 9 divides n^2m.
(2) If 9 divides m, then since m divides n^2m, it follows from what we proved in class that 9 divides n^2m.

In any case 9 divides n^2m. (3) Let a, b be integers with $b \neq 0$. Prove that any integer

solution to the quadratic equation $x^2 + ax + b = 0$ divides b.

Solution. Suppose z is an integer solution to $x^2 + ax + b = 0$. WTP z divides b. By our assumption, $z^2 + za + b = 0$, hence $b = -z^2 - za = z(-z - a)$. Since $-z - a \in \mathbb{Z}$, it follows that z divides b. (4) Let x, y be real numbers. Prove that if $0 < x < y$ then there

is $\epsilon > 0$ such that $\sqrt{x^2 + \epsilon} < y$.

Solution: Suppose that $x, y \in \mathbb{R}$ and $0 < x < y$. WTP $\exists \epsilon > 0$, $\sqrt{x^2 + \epsilon} < y$.

(The next step of the proof is to define ϵ as we are in an existential proof so we do a side computation: $\sqrt{x^2 + \epsilon} < y$
$x^2 + \epsilon < y^2$ $\epsilon < y^2 - x^2$ so we can choose $\epsilon = \frac{y^2 - x^2}{2}$) Back to the proof:

Define $\epsilon = \frac{y^2 - x^2}{2}$. Since $x < y$, and x, y are positive, $x^2 < y^2$ and therefore $y^2 - x^2 > 0$ and also $\epsilon = \frac{y^2 - x^2}{2} > 0$. Let us show that $\sqrt{x^2 + \epsilon} < y$. Indeed

$$x^2 + \epsilon = x^2 + \frac{y^2 - x^2}{2} < x^2 + (y^2 - x^2) = y^2$$

Since y is positive, it follows that $\sqrt{x^2 + \epsilon} < \sqrt{y^2} = y$.