Math 300 Intro Math Reasoning
Worksheet 02: Mathematical logic

(1) Consider the statement:

$\alpha =$"Every real solution of $x^2 + x - 6 = 0$ is positive."

(1) Formalize it using the predicate calculus. **Solution** $\forall x \in \mathbb{R}(x^2 + x - 6 = 0 \Rightarrow x > 0)$
(2) Give examples of sets of discourse A, B such that α is true in A and α is false in B.

Solution: If $A = [0, \infty)$, then α is true in A since we only range on non-negative x's and if $x^2 + x - 6 = 0$ then $x = 3 > 0$. If $B = \mathbb{R}$ then α is false since for example $x = -2$ is a solution to the equation which is negative.

(2) Write the negation of the following sentence **without** the negation symbol "$\neg$" and determine whether it is true or false in the set $\mathbb{R}$:

"$(\exists x(x > 5)) \Rightarrow (\forall y(y > -100))$."

Solution: $\sim ((\exists x(x > 5)) \Rightarrow (\forall y(y > -100))) \equiv \exists x(x > 5) \wedge \exists y(y \leq -100)$. The negation is true.

(3) What are all the $x \in \mathbb{N}$ such that $\exists y,\ x + y = 4$?

Solution: $\{0, 1, 2, 3, 4\}$.

(4) Show that the following are not logically equivalent:

$$\forall x, \exists y, P(x, y) \text{ and } \exists y, \forall x, P(x, y)$$

Hint: Find a domain and interpretation for $P(x, y)$ under which one of the formulas is true and the other is false.

Solution: Let $P(x, y)$ be $x < y$. Then in the domain of $\mathbb{Z}$, we have that for every number $x \in \mathbb{Z}$ there is a number $y \in \mathbb{Z}$ such that $x < y$ (for example we can take $y = x + 1$). On the other hand, there cannot be a single $y \in \mathbb{Z}$ such that for every $x \in \mathbb{Z}$, $x < y$, because such a number y would be the largest integer (and for example y would not be less than y).