Filters and Ultrafilters- a mathematical approach to smallness and largeness

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Examples

Classical notions of and "large":

1. A large enough number - where is the cut-off?
2. Infinite sets.
3. Large cardinality (for example, uncountable sets).
4. Events of probability one/positive.
5. Full Lebesgue measure.

Classical notions of and "small":

1. a small number.
2. finite sets.
3. sets of small cardinality.
4. events of probability zero.
5. Lebesgue measure 0.

You can clearly see that there is some Duality here between small and large.

Q. Is there a single definition that treats all these notions?

Filters

Definition 1

Let A be any set. A family of sets $U \subseteq P(A)$ is called a *filter* over A if:

1. $A \in U$, $\emptyset \notin U$.
2. U is closed under intersections: for all $X, Y \in U$, $X \cap Y \in U$.
3. U is upward closed: for all $X \in U$ an for every $X \subseteq Y \subseteq A$, $Y \in U$.

Important examples of filters:

- The Freće filter on any infinite cardinal κ is $\mathcal{FR}_\kappa = \{X \subseteq \kappa \mid |\kappa \setminus X| < \kappa\}$.
- Given a probability space $(\Omega, \mathbb{P}, \Sigma)$, let $\mathcal{F}_0 = \{X \in \Sigma \mid \mathbb{P}(X) = 1\}$.
- The filter of measurable sets with null complements $\mathfrak{L} = \{X \subseteq \mathbb{R} \mid \mu_{Leb}(\mathbb{R} \setminus X) = 0\}$.
- Given a family of non-empty sets $\mathcal{B} \subseteq P(A)$, closed under finite intersection, $\mathcal{F}_\mathcal{B} = \{X \subseteq A \mid \exists b \in \mathcal{B},\ b \subseteq X\}$ is a called, the filter generated by X. A similar construction works if $\mathcal{B}$ only has the finite intersection property.

We think of sets in a filter as large sets, and a fixed filter is a fixed notion of largeness.

Ideals

The dual notion of a filter, which corresponds to smallness is the notion of ideals:

Definition 2

Let A be any set. A family of sets $\mathcal{I} \subseteq P(A)$ is called a *ideal* over A if:

1. $\emptyset \in U$, $A \notin U$.
2. U is closed under unions.
3. U is downward closed to inclusion.

- The Frećhe ideal on any infinite cardinal κ is $\mathcal{I}_\kappa = \{X \subseteq \kappa \mid |X| < \kappa\}$.
- Given a probability space $(\Omega, \mathbb{P}, \Sigma)$, let $\mathcal{I}_0 = \{X \in \Sigma \mid \mathbb{P}(X) = 0\}$.
- The filter of measurable sets with null complements $\mathfrak{L} = \{X \subseteq \mathbb{R} \mid \mu_{Leb}(X) = 0\}$.
- Given a filter $\mathcal{F}$ over A, the *dual ideal*, denoted by $\mathcal{F}^* = \{A \setminus X \mid X \in \mathcal{F}\}$.

This is a specific case of the other notion of ideals from algebra (where product is intersection and addition is union).

Ultrafilters

Given a filter $\mathcal{F}$, we have:

- Sets in $\mathcal{F}$ are large.
- Sets in $\mathcal{F}^*$ are small.
- What about sets which are neither in $\mathcal{F}$ nor in $\mathcal{F}^*$? For example, if $\mathcal{F} = \mathcal{FR}_\omega$ the sets $X \subseteq \omega$ such that $|X| = |\omega \setminus X| = \aleph_0$.

Given an ideal I, a set $X \notin I$ is called positive.

Definition 3

A filter U over A is called an *ultrafilter* if:

4. for every $X \subseteq A$, either $X \in U$ or $A \setminus X \in U$.

So an ultrafilter U determines whether each set is small or large. In other words, all positive sets are large.

Example 4

Given $a \in A$, define $U_a = \{X \subseteq A \mid a \in X\}$. Then U_a is called a principal ideal.

Q. Are there any non-principal ultrafilters?

Non-principal ultrafilters

Exercise 1

If U is an ultrafilter over a finite set X then U is principal.

Theorem 5 (The ultrafilter lemma)

Assume AC. Then every filter can be extended to an ultrafilter. Namely each filter $\mathcal{F}$ is a subset of some ultrafilter U.

Applying the previous Theorem to $\mathcal{FR}_\omega = \{X \subseteq \omega \mid |\omega \setminus X| < \omega\}$, we obtain an ultrafilter U. It cannot be principle, since if $\{a\} \in U$, then so is $\omega \setminus \{a\} \in \mathcal{FR}_\omega \subseteq U$, but then $\emptyset \in U$, contradiction.
Ultrafilters are highly non-constructive sets. Every $X \in P(\omega)$ can be identified with a real number $0.f_X(0)f_X(1)f_X(2)... \in [0,1]$ where f_X is the indicator function. Under this identification, an ultrafilter can be identified with a subset of $[0,1]$.

Theorem 6

If U is a non-principle ultrafilter over ω, then U is not Lebesgue measurable when identified as a subset of $[0,1]$.

Part 2: Applications

Arrow's Theorem

Suppose that in an election there are finitely many $n(\geq 3)$ candidates $\{c_1, ..., c_n\}$ and a set X of voters. Each voter makes a ranking(ordering of $c_1, ..., c_n$) of the candidates. An *fair election system*, is a way of configuring from each possible ranking list of the voters, an outcome of the election (a single ranking) which conforms to the following two rules:

1. if all the voters enter the same ranking, then this is the outcome;
2. whether a candidate a precedes candidate b in the outcome depends only on their order on the different ranking lists of the individual voters (and it does not depend on where a and b are on those lists; i.e., on how the voters ranked other candidates).

Formally, an election system is a function $G : S_n^X \to S_n$ (where S_n is the set of permutations on n elements).

Theorem 7 (Arrow's Theorem)

If G is a fair election system on a finite set of voters, then there is a dictator, namely, there is $x_{dictator} \in X$ such that for every possible input $(\sigma_x)_{x\in X} \in S_n^X$, $G((\sigma_x)_{x\in X}) = \sigma_{x_{dictator}}$.

Arrow's Theorem on an arbitrary set of voters

When X is an arbitrary set of voters we have the following generalization:

Theorem 8

If G is a fair election system, then there is an ultrafilter H on X such that for any input $(\sigma_x)_{x \in X} \in S_n^X$, $G((\sigma_x)_{x \in X}) = \pi$ if and only if the set $\{x \in X \mid \sigma_x = \pi\} \in H$.

In particular, we get Arrow's original theorem since ultrafilters on finite sets are principal.

The ultraproduct construction

Suppose that $\mathcal{M} = (M_i)_{i \in I}$ is an I-indexed family of mathematical objects of the same type (e.g. groups, topological spaces, vector spaces, or in general models over the same language). Let U be an ultrafilter over I. Using U, we can integrate $\mathcal{M}$ into an object as follows:
Consider $\prod_{i \in I} M_i = \{f \mid Dom(f) = I \wedge f(i) \in M_i\}$. Define an equivalence relation

$$f \sim_U g \Longleftrightarrow \{i \in I \mid f(i) = g(i)\} \in U$$

The ultraproduct is defined to be the set $\prod_{i \in I} M_i/U = \{[f]_{\sim_U} \mid f \in \prod_{i \in I} M_i\}$.

Example 9

Suppose that each $M_i = (G_i, e_i, *_i)$ is a group. Then define over $G = \prod_{i \in I} G_i/U$ $e = [i \mapsto e_i]_U$ and $[f]_U * [g]_U = [i \mapsto f(i) *_i g(i)]_U$. One can check that $(G, e, *)$ is also a group.

If every $M_i = M$ then we call this construction the ultrapower and denote it by M^I/U.

Theorem 10 (Lós Theorem for groups)

Suppose that $\phi(x_1, ..., x_n)$ *is a formula in the language of groups* $\mathfrak{L}_G = \{e, *, ^{-1}\}$ *Then*

$$(G, e, *) \models \phi([f_1]_U, ..., [f_n]_U) \text{ iff } \{i \in I \mid (M_i, e_i, *_i) \models \phi(f_1(i), ..., f_n(i))\} \in U$$

So, for example, suppose that G is a group, and $\mathcal{S}$ is a system of equations (might be infinite) such that every finite subset of equations of S has a solution in G, then there is an extension of G to a group $G \subseteq G'$ where in G' S has a solution.

The compactness theorem

Theorem 11

Suppose that T is a theory in the language $\mathcal{L}$ such that every finite set of sentences in T is consistent. Then T is consistent.

Let $(X, \tau_X), (Y, \tau_Y)$ be Hausdorff topological spaces. Recall that

Definition 12

A function $f : X \to Y$ is continuous in the sequential sense if whenever $(x_n)_{n=0}^{\infty} \subseteq X$ is a sequence converging to $x \in X$ (namely, for every neighborhood $x \in U \in \tau_X$ there is N such that for all $n \geq N$, $x_n \in U$), the sequence $(f(x_n))_{n=0}^{\infty}$ converges to $f(x)$.

It is well known that first-countable spaces a function f is continuous if and only if f is continuous in the sequential sense. In general the two are not equivalent (For example $f : \omega_1 + 1 \to \mathbb{R}$ defined by $f(x) = 0$ if $x < \omega_1$ and $f(\omega_1) = 1$ is not continuous but sequentially continuous.)

Definition 13

A *net* is a function $\vec{x} = (x_a)_{a \in A}$ such that $(A, \leq_A)$ is a directed set. x is a limit of $\vec{x}$ if for every $x \in U \in \tau_X$ there is a such that , $b \geq a$, $x_b \in U$ (AKA Moore-Smith convergence).

Now a function $f : X \to Y$ is continuous iff for every net $(x_a)_{a \in A}$ with limit x, $(f(x_a))_{a \in A}$ has limit $f(x)$.

Cofinal maps

Some "types" of directed sets actually give essentially the same notion of net, for example, $\mathbb{N}$ and $\mathbb{N}_{even}$ or even $fin = \{X \in P(\mathbb{N}) \mid X \text{ is finite}\}$. More generally we would like to find an equivalence relation that reduces to the "essential" ordered sets. This is given by the Tukey order which was defined by J. Tukey [7]:

Definition 14

Let $(P, \leq_P), (Q, \leq_Q)$ be two partially ordered (directed) sets. Define $(P, \leq_P) \leq_T (Q, \leq_Q)$ iff there is a cofinal map[a] $f : Q \to P$. Define $(P, \leq_P) \equiv_T (Q, \leq_Q)$ iff $(P, \leq_P) \leq_T (Q, \leq_Q)$ and $(Q, \leq_Q) \leq_T (P, \leq_P)$.

[a] if for every cofinal $B \subseteq Q$, $f[B] \subseteq P$ is cofinal.

If $B \leq_T A$, then any B-net $(x_b)_{b\in B}$ can be now replaced by $(x_{f(a)})_{a\in A}$ and if x is a limit point of $(x_b)_{b\in B}$ then x must be a limit of $(x_{f(a)})_{a\in A}$.
The research of what are the "essential" A's is a completely set theoretic (order theoretic) question.

Classic results of Todorcevic

Theorem 15 (Todorcevic 85[6])

Assuming $MA_{\aleph_1}$ it is consistent that there are exactly 5 Tukey classes of directed posets of cardinality at most $\aleph_1$.

Theorem 16 (Todorcevic 85[6])

for any regular $\kappa > \omega$, there are 2^κ-many distinct Tukey classes of cardinality $\kappa^{\aleph_0}$. In particular, there are at least $2^{cf(\mathfrak{c})}$ many distinct Tukey classes of cardinality $\mathfrak{c}$.

Definition 17

Given a net $\vec{x} = (x_a)_{a \in A}$, define for each $a \in A$, $x_{\geq a} = \{x_b \mid b \geq a\}$. The filter associated with $\vec{x}$, denoted by $F_{\vec{x}}$ is the filter generated by the sets $x_{\geq a}$. Namely, $T \in F_{\vec{x}}$ iff $\exists a \in A,\ x_{\geq_a} \subseteq T$.

Indeed, $F_{\vec{x}} \subseteq P(X)$ is a filter over X. The filter $F_{\vec{x}}$ determines the convergence properties of the net $\vec{x}$ in the sense that $\vec{x}$ converges to x iff $\mathcal{N}(x) \subseteq F_{\vec{x}}$. This gives rise to the idea of converging filters:

Definition 18

We say that a filter F converges to a point x if $\mathcal{N}(x) \subseteq F$.

Since every filter can be extended to an ultrafilter, if F converges to a point x then there is an ultrafilter which covergese to x as well. Therefore, for most purposes, it suffices to consider only ultrafilters, or *ultranets*. For example, TFRE:

- $f : X \to Y$ is continuous.
- For every $x \in X$, and every ultrafilter U such that $\mathcal{N}(x) \subseteq U$, the ultrafilter $f_*(U) = \{B \subseteq Y \mid f^{-1}[B] \in U\}$ extends $\mathcal{N}(y)$.

The Tukey order on ultrafilters

As we have seen earlier, it suffices to study the cofinal types of ultrafilters. This motivates the study of the directed order $(U, \supseteq)$ where U is an ultrafilter.

Proposition 1

Suppose that $U \leq_T V$ where U, V are ultrafilters, then there is a (weakly) monotone map $f : V \to U$ which is cofinal.

The Tukey order has been studied extensively on ultrafilters on ω by Blass, Dobrinen, Milovic, Raghavan, Shelah, Solecki, Todorcevic, Verner and many others. It still entails quite challenging open problems.

A taste from my research: The Tukey class of a Fubini product of ultrafilters.

Fact 19

Let $(P, \leq_P), (Q, \leq_Q)$ *be directed orders. Then*[a] $(P \times Q, \leq_\times)$ *is the least upper bound of* P, Q *in the Tukey order. Hence* $P =_T P \times P$.

[a] $(p, q) \leq_\times (p', q')$ if and only if $p \leq_P p'$ and $q \leq_Q q'$.

Definition 20 (Fubini product)

Suppose that U is a ultrafilter over X and V an ultrafilter over Y. We denote by $U \cdot V$ the Fubini product of U and V which is the ultrafilter defined over $X \times Y$ as follows, for $A \subseteq X \times Y$,

$$A \in U \cdot V \text{ if and only if } \{x \in X \mid (A)_x \in Y\} \in U$$

where $(A)_x = \{y \in Y_x \mid \langle x, y \rangle \in A\}$. If $U = V$, then U^2 is defined as $U \cdot U$ and refered to as the Fubini power.

It is not hard to check that $U \cdot V$ is also an ultrafilter and to show that $(U, \supseteq), (V, \supseteq) \leq_T (U \cdot V, \supseteq)$. Therefore $(U \times V, \leq_\times) \leq_T (U \cdot V, \supseteq)$.

Theorem 21 (Dobrinen-Todorcevic-Milovich)

For any U, V, $U \cdot V =_T U \times \prod_{n<\omega} V$.

Definition 22

Let U be an ultrafilter over $\mathbb{N}$.

- U is a p-point if every sequence $\langle X_n \mid n < \omega \rangle \subseteq U$ has a U-measure one pseudo intersection.
- U is rapid if for every function $f : \mathbb{N} \to \mathbb{N}$ there is $X \in U$ such that for every $n < \omega$, $X(n) \geq f(n)$.

These definitions are obviously generalized to any cardinal $\kappa > \omega$.

Theorem 23 (Dobrinen-Todorcevic[3])

Suppose that V, U are ultrafilters on ω, V is a rapid p-point. Then $U \cdot V \equiv_T U \times V$. In particular, if U, V are rapid p-points then $U \cdot V =_T V \cdot U$.

In particular if U is a rapid p-point then $U \cdot U \equiv_T U$. Moreover, Dobrinen and Todorcevic constructed an example of a non-rapid p-point ultrafilter U such that $U <_T U^2$.

Theorem 24 (Milovich[5])

If U is a p-point ultrafilter then on ω and V is any ultrafilter, then $V \cdot U = V \times U \times \omega^\omega$ and therefore if U, V are both p-points then $U \cdot V =_T V \cdot U$.

Theorem 25 (Dobrinen-B.[2])

Let U, V be any κ-complete ultrafilters over $\kappa > \omega$, then $U \cdot V \equiv_T U \times V$. In particular $U \cdot V =_T V \cdot U$ and $U \cdot U \equiv_T U$.

Theorem 26 (B. [1]2024)

For any two ultrafilters U, V (on any cardinal), $U \cdot V =_T V \cdot U$.

Thank you for your attention!

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Trying to relax the assumption $\kappa^{<\kappa} = \kappa$ in Gavin's theorem, we have the following consistency result by Abraham and Shelah.

Theorem 27 (Abraham-Shelah forcing)

Assume GCH, let κ be a regular cardinal, and $\kappa^+ < cf(\lambda) \leq \lambda$. Then there is a forcing extension by a κ-directed, cofinality preserving forcing notion such that $2^{\kappa^+} = \lambda$ and there is a sequence $\langle C_i \mid i < \lambda \rangle$ such that:

1. *C_i is a club at κ^+.*
2. *for every $I \in [\lambda]^{\kappa^+}$, $|\bigcap_{i \in I} C_i| < \kappa$.*

In particular, $\neg Gal(Cub_{\kappa^+}, \kappa^+, 2^{\kappa^+})$.

A natural question is what happens on inaccessible cardinals? of course, by Galvin's theorem, we should be interested in weakly inaccessible Cardinals.

Question

Is it consistent to have a weakly inaccessible cardinal κ such that $\neg Gal(Cub_\kappa, \kappa, \kappa^+)$?

There are some limiting results due to Garti (see [4])

At successors of singular cardinals

Our focus is on the second case which does not fall under Abraham-Shelah's- the case of successors of singulars. Is it consistent to have $\neg Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$ for a singular κ? Again, by Galvin's theorem, this would require violating SCH.

Theorem 28 (Garti, Poveda and B.)

Assume GCH and that there is a (κ, κ^{++})-extender[a]. Then there is a forcing extension where $cf(\kappa) = \omega$ and $\neg Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$.

[a] This situation can be forced just from the assumption $o(\kappa) = \kappa^{++}$

The idea is to Easton-support iterate the Abraham-Shelah's forcing on inaccessibles $\leq \kappa$. This produces a model of $\neg Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$. Using a sophistication of Woodin's argument due to Ben-Shalom [**?**], we can argue that κ remains measurable after this iteration. Finally, singularize κ using Prikry/Magidor forcing. The key lemma is the following:

Lemma 29

A κ^+-cc forcing preserves a witness for $\neg Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$.

The strong negation at successor of singulars

The sequence of clubs $\langle C_i \mid i < \kappa^+ \rangle$ produced by the Abraham-Shelah forcing, witnesses a stronger failure of $Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$, indeed for any $I \in [\kappa^{++}]^{\kappa^+}$, $\cap_{i \in I} C_i$ is actually of **size less than** κ. Let us denote this by $\neg_{st} Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$.
Interestingly, the previous argument does work for the strong negation:

Proposition 2

In general κ^+-cc forcings do not preserve $\neg_{st} Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$.

Indeed, any forcing which adds a set of size κ which diagonalize $(Cub_\kappa)^V$ (e.g. diagonalizing the club filter, Magidor forcing with $o(\kappa) \geq \kappa$) kills $\neg_{st} Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$ (namely satisfy $\neg(\neg_{st} Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})))$.

Question

Is it a ZFC-theorem that $\neg_{st} Gal(Cub_{\kappa^+}, \kappa^+, \kappa^{++})$ cannot hold at a successor of a singular cardinal? Explicitly, is it true that from any sequence of κ^{++}-many clubs at κ^+ one can always extract a subfamily of size κ^+ for which the intersection is of size at least κ?

Two opposite results for Prikry forcing

On one hand Prikry forcing does not add a set of cardinality κ which diagonalize $(Cub_\kappa)^V$:

Theorem 30

Let U be a normal ultrafilter over κ. Let $\langle c_n \mid n < \omega \rangle$ be V-generic Prikry sequence for U, and suppose that $A \in V[\langle c_n \mid n < \omega \rangle]$ diagonalize $(Cub_\kappa)^V$. Then, there exists $\xi < \kappa$ such that $A \setminus \xi \subseteq \{c_n \mid n < \omega\}$. In particular, $|A \setminus \xi| \leq \aleph_0$.

On the other hand, just forcing a Prikry sequence is not enough:

Theorem 31

Let $\mathcal{C}$ be a witness for the strong negation. Then there exists $\mathcal{D}$, such that:

1. *$\mathcal{D}$ is also a witness for the strong negation;*
2. *For every normal ultrafilter U over κ, forcing with Prikry(U) yields a generic extension where $\mathcal{D}$ cease to be a witness.*