Introduction to advanced mathematics-Solutions to "More Exercises"

November 30, 2022

Problem 1. Compute the following sets, prove your answer:

1. $\{n \in \mathbb{Z} \mid (-5) \cdot n < n\}$. **solution** $\mathbb{N}_+$ 0,1,2,3,4,1/2
n+y^2=-1
2. $\{x \in \mathbb{R} \mid \exists y \in \mathbb{R}. \exists n \in \mathbb{N}. n + y^2 = x\}$ **solution** $[0, \infty)$
3. $\{X \cup \{0\} \mid X \in P(\mathbb{N})\}$. **solution** $P(\mathbb{N}) \setminus P(\mathbb{N}_+)$
4. $\{X \in P(\mathbb{Q}) \mid X \cup \mathbb{N} \subseteq \mathbb{Z}\}$ **solution** $P(\mathbb{Z})$
5. $\{x \in \mathbb{R} \mid |[x, x+1] \cap \mathbb{Z}| < 2\}$ **solution** $\mathbb{R} \setminus \mathbb{Z}$.

Problem 2. Prove or disprove the following statements:

1. If $A = A \setminus B$ then $B = \emptyset$.
solution Disprove, $A = \{1,2\}$ $B = \{3\}$, we have $\{1,2\} = \{1,2\} \setminus \{3\}$ but also $\{3\} \neq \emptyset$

2. If $A = A \setminus B$ then $A \cap B = \emptyset$.
solution Proof, Suppose that $A = A \setminus B$, we want to prove that $A \cap B = \emptyset$. Suppose toward a contradiction that $A \cap B \neq \emptyset$, then there is $x \in A \cap B$. By definition of intersection, $x \in A$ and $x \in B$. It follows that $x \ni nA \setminus B$. Hence x is a member of A which is not a member of $A \setminus B$, contradiction the assumption $A = A \setminus B$.

3. If $A \cup B = A \cup C$ and $A \cap B = A \cap C$ then $B = C$. **solution:** Prove, auppose that $A \cup B = A \cup C$ and $A \cap B = A \cap C$, we want to prove that $B = C$. Let us prove it by a double inclusion:

- $\underline{B \subseteq C}$ Let $x \in B$, we want to prove that $x \in C$. Let us split into cases:
  (a) If $x \in A$ then $x \in A \cap B$ and by the assumption that $A \cap B = A \cap C$ it follows that $x \in A \cap C$ and in particular $x \in C$.
  (b) If $x \notin A$, recall that $x \in B$ and therefore $x \in A \cup B$. By the assumption $x \in A \cup C$ and since $x \notin A$ it follows that $x \in C$.
- $\underline{C \subseteq B}$ Symmetric to the first inclusion.

4. If $A \Delta C \subseteq A \Delta B$ then $A \cap C \subseteq B$.

   solution Disprove, Take $A = \{1,2\}$ $C = \{1\}$ $B = \{3\}$ then $A \Delta C = \{2\} \subseteq \{1,2,3\} = A \Delta B$ but $A \cap C = \{1\}$ is not a subset of $B = \{3\}$.

5. $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.

   solution Prove.

Problem 3. Let A, B, C, D be sets. Prove that

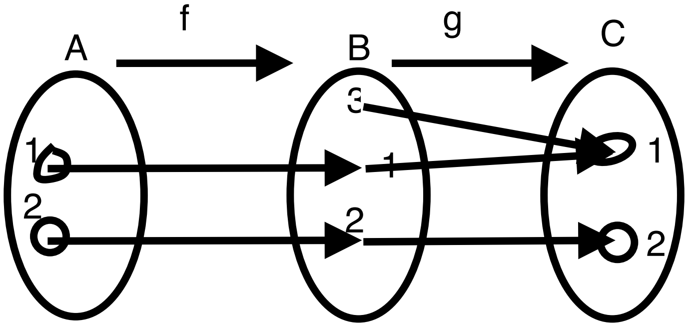
$$(A \times B) \setminus (C \times D) = [(A \setminus C) \times B] \cup [A \times (B \setminus D)]$$

Problem 4. Prove the for any sets A, B:

$$A \times B = B \times A \Leftrightarrow [A = B \vee A = \emptyset \vee B = \emptyset]$$

Proof. We need to prove a double implication.

- $\underline{\Rightarrow}$ Suppose that $A \times B = B \times A$, and suppose toward a contradiction that $A \neq B$, and $A, B \neq \emptyset$ let us split into cases:
  - If there is $x \in A$ such that $x \notin B$, take any $b \in B$, then $\langle x, b \rangle \in A \times B$ but $\langle x, b \rangle \notin B \times A$ since $x \notin B$.
  - the case that there is $x \in B$ such that $x \notin A$ is symmetric.
- $\underline{\Leftarrow}$ Suppose that $A = B \vee A = \emptyset \vee B = \emptyset$ and we want to prove that $A \times B = BA$. Let us split into cases:
  - If $A = B$ then $A \times B = A \times A = B \times A$.
  - If $A = \emptyset$ then $A \times B = \emptyset = B \times A$
  - the case $B = \emptyset$ is similar.



□

Problem 5. Let A and B be any sets. Prove that:

1. $P(A \cap B) = P(A) \cap P(B)$.
2. $P(A \cup B) = P(A) \cup P(B)$ if and only if $A \subseteq B \vee B \subseteq A$.
3. $P(A \setminus B) \subseteq \{\emptyset\} \cup (P(A) \setminus P(B))$
4. If $P(A) \subseteq P(A \setminus B)$ then $A \cap B = \emptyset$.

Problem 6. Let $f : A \to B$ and $g : B \to C$ be function. Prove or disprove the following statements:

1. If $g \circ f$ is injective the g is injective.
   Solution Disprove. For example, take $f : \{1, 2\} \to \{1, 2\}$, $f(x) = x$ (the identity and $g : \{1, 2, 3\} \to \{1, 2, 3\}$ defined by $g(1) = 1, g(2) = 2, g(3) = 2$ clearly g is not injective and $g \circ f : \{1, 2\} \to \{1, 2\}$ $g \circ f(1) = 1$ and $g \circ f(2) = 2$, thus $g \circ f$ is injective.

2. If $g \circ f$ is injective the f is injective.
   Solution Prove. Suppose that $g \circ f$ is injective and assume towards a contradiction that f is not injective. Then there are $a_1, a_2 \in A$ distinct such that $f(a_1) = f(a_2) = b$. In particular,

   $$g \circ f(a_1) = g(f(a_1)) = g(b) = g(f(a_2)) = g \circ f(a_2)$$

   contradiction the assumption that $g \circ f$ is injective.

3. If $g \circ f$ is surjective then f is surjective.
   Solution Disprove.

4. If $g \circ f$ is surjective the g is surjective.
   Solution Prove.

5. If f is surjective and g is not injective then $g \circ f$ is not injective.
   Solution: Suppose that g is not injective and f is surjective. We want to prove that $g \circ f$ is not injective. Let $b_1, b_2 \in B$ be distinct such that $g(b_1) = g(b_2)$. Since f is surjective, then there are a_1 and a_2 such that

$f(a_1) = b_1$ and $f(a_2) = b_2$. Note that $a_1 \neq a_2$ since f is a function (otherwise, $a_1 = a_2 = a$ and $f(a) = b_1$ and $f(a) = b_1$). It follows that

$$g \circ f(a_1) = g(f(a_1)) = g(b_1) = g(b_2) = g(f(a_2)) = g \circ f(a_2)$$

Hence $g \circ f$ is not injective.

Problem 7. Determine if the following functions are injective/surjective/bijective. If the function is invertible, compute its inverse.

1. $f : \mathbb{N} \longrightarrow \mathbb{N}$, $f(n) = n^2 - n + 2$.
   Solution: Not injective: for example $f(0) = 2 = f(1)$.
   Not surjective: for example $0 \notin \mathbb{N}$, to see this, we have already seen that $f(0), f(1) \neq 0$, for $n \geq 2$, $f(n) = n^2 - n + 2 = n(n-1) + 2 \geq 2 > 0$.

2. $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \begin{cases} 0 & x = 1 \\ \frac{1}{x-1} & x \neq 1 \end{cases}$.

   solution Injective: Let $x_1, x_2 \in \mathbb{R}$. Suppose that $f(x_1) = f(x_2)$, we want to prove that $x_1 = x_2$. Note that if $x \neq 1$, the $f(x) = \frac{1}{x-1} \neq\neq 0$. Split into cases:

   - If $f(x_1) = f(x_2) = 0$, then as we have seen $x_1 = x_2 = 1$.
   - IF $f(x_1) = f(x_2) \neq 0$, then $x_1, x_2 \neq 1$ and therefore $f(x_1) = \frac{1}{x_1 - 1} = \frac{1}{x_2 - 1} = f(x_2)$. From simple algebra we get $x_1 = x_2$.

   Surjective: Let $y \in \mathbb{R}$, we want to prove that there is $x \in \mathbb{R}$ such that $f(x) = y$. If $y = 0$, define $x = 1$, then by definition $f(1) = 0$. If $y \neq 0$ define $x = \frac{1}{y} + 1$, then $x \neq 1$ and we have

   $$f(x) = \frac{1}{x-1} = \frac{1}{(\frac{1}{y} + 1) - 1} = \frac{1}{\frac{1}{y}} = y$$

   By the theorem we have seen in class, f is invertible and the function $f^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$ defined by

   $$f^{-1}(y) = \begin{cases} 1 & y = 0 \\ \frac{1}{y} + 1 & else \end{cases}$$

3. $f : \mathbb{N} \to P(\mathbb{N})$, $f(n) = \{k \in \mathbb{N} \mid k < n\}$.

   Solution Injective: let $n_1, n_2 \in \mathbb{N}$, we suppose that $n_1 \neq n_2$, we want to prove that $f(n_1) \neq f(n_2)$. Suppose that $n_1 < n_2$ (the case $n_2 < n_1$ is symmetric), we want to prove $\{k \in \mathbb{N} \mid k < n_1\} \neq \{k \in \mathbb{N} \mid k < n_2\}$. Note that $n_1 < n_2$, and by the separation principle, $n_1 \in \{k \in \mathbb{N} \mid k < n_2\}$ but $n_1 \not< n_1$ so $n_1 \notin \{k \in \mathbb{N} \mid k < n_1\}$. So we found an element $n_1 \in f(n_2)$ such that $n_1 \notin f(n_1)$, and in particular $f(n_1) \neq f(n_2)$.
   Not surjective: For example $\{1\} \in P(\mathbb{N})$, suppose toward a contradiction that there is n such that $f(n) = \{1\}$, then $\{k \in \mathbb{N} \mid k < n\} = \{1\}$. By set equality $1 \in \{k \in \mathbb{N} \mid k < n\}$ and by separation $1 < n$. It follows that $0 < n$ and therefore $0 \in \{k \in \mathbb{N} \mid k < n\}$. By the list principle, $0 \notin \{1\}$ which contradicts the equality $f(n) = \{1\}$.

4. $f : \mathbb{N} \times \mathbb{N} \to P(\mathbb{N})$, $f(\langle n, m\rangle) = \{n, m\}$. **Solution** Not injective, not surjective.

5. $f : \mathbb{N} \times \mathbb{N} \to P(\mathbb{N})$, $f(\langle n, m\rangle) = \{n, n + m\}$ **Solution** Injective, not surjective.

6. $f : P(\mathbb{N}) \to P(\mathbb{N}_{even}) \times P(\mathbb{N}_{odd})$, $f(X) = \langle X \cap \mathbb{N}_{even}, X \cap \mathbb{N}_{odd}\rangle$. **Solution** Injective and surjective. The inverse function is $f^{-1} : P(\mathbb{N}_{even}) \times P(\mathbb{N}_{odd}) \to P(\mathbb{N})$, defined by $P(\langle X, Y\rangle) = X \cup Y$.

Problem 8. Prove by induction the following claims:

- For every $n \geq 1$,

  $$2 + 4 + 6 + \cdots + 2n = n(n + 1)$$

  Solution

  - **Induction basis:** For $n = 1$ we need to prove that $2 = 1 \cdots 2$ which is clear.
  - **Induction hypothesis**: Suppose that

    $$2 + 4 + 6 + \cdots + 2n = n(n + 1)$$

    for a general n

- **Induction step;** We want to prove

$$2 + 4 + 6 + \cdots + 2(n + 1 = (n + 1)(n + 2)$$

Indeed

$$2+4+6+\cdots+2n+2(n+1) = n(n+1)+2(n+1) = (n+2)(n+2)$$

- For any $n \geq 1$,

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + (2n + 1) \cdot 2^{2n+1} = 2 + n \cdot 2^{2n+3}$$

Solution

- **Induction basis:** We need to prove that

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 = 2 + 1 \cdot 2^{2+5}$$

Indeed $2 + 8 + 24 = 34 = 2 + 32 = 2 + 2^5$.

- **Induction hypothesis** Assume

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + (2n + 1) \cdot 2^{2n+1} = 2 + n \cdot 2^{2n+3}$$

For a general n.

- **Induction step:** We want to prove that

$$1{\cdot}2^1{+}2{\cdot}2^2{+}3{\cdot}2^3{+}\ldots{+}(2n{+}1){\cdot}2^{2n+1}{+}(2n{+}2)2^{2n+2}{+}(2n{+}3)2^{2n+3} = 2{+}(n{+}1){\cdot}2^{2n+5}$$

We have

$$1 \cdot 2^1 + \ldots + (2n + 1) \cdot 2^{2n+1} + (2n + 2)2^{2n+2} + (2n + 3)2^{2n+3} =$$

$$= 2{+}n2^{2n+3}{+}(2n{+}2)2^{2n+2}{+}(2n{+}3)2^{2n+3} = 2{+}2^{2n+2}(2n{+}2n{+}2{+}2(2n{+}3) =$$
$$= 2 + 2^{2n+2}(8n + 8) = 2 + (n + 1) \cdot 2^{2n+5}$$

- For any $n \geq 1$,

$$\frac{3}{2} + \frac{9}{4} + \frac{33}{8} + \ldots + \frac{2^{2n-1} + 1}{2^n} = \frac{2^{2n} - 1}{2^n}$$

Solution

- **Induction basis:** We need to prove that $\frac{3}{2} = \frac{2^2-1}{2^2}$ which is true.
- **Induction hypothesis** Assume

$$\frac{3}{2} + \frac{9}{4} + \frac{33}{8} + \dots + \frac{2^{2n-1}+1}{2^n} = \frac{2^{2n}-1}{2^n}$$

for a general n.
- **Induction step;** We want to prove

$$\frac{3}{2} + \frac{9}{4} + \frac{33}{8} + \dots + \frac{2^{2n-1}+1}{2^n} + \frac{2^{2n+1}+1}{2^{n+1}} = \frac{2^{2n+2}-1}{2^{n+1}}$$

Indeed

$$\frac{3}{2} + \frac{9}{4} + \frac{33}{8} + \dots + \frac{2^{2n-1}+1}{2^n} + \frac{2^{2n+1}+1}{2^{n+1}} =$$

$$= \frac{2^{2n}-1}{2^n} + \frac{2^{2n+1}+1}{2^{n+1}} =$$

$$= \frac{2 \cdot 2^{2n} - 2 + 2^{2n+1} + 1}{2^{n+1}} = \frac{2^{2n+2}-1}{2^{n+1}}$$

- For any $n \geq 1$,

$$\frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \dots + \frac{1}{(2n-1)2n} = \frac{1}{2n}$$

Solution

- **Induction basis:** For $n = 1$, $\frac{1}{1\cdot(1+1)} = \frac{1}{2\cdot 1}$.
- **Induction hypothesis** Assume

$$\frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \dots + \frac{1}{(2n-1)2n} = \frac{1}{2n}$$

for a general n.
- **Induction step:** We want to prove that

$$\frac{1}{(n+1)(n+2)} + \dots + \frac{1}{(2n+1)(2n+2)} = \frac{1}{2(n+1)}$$

We add and subtract $\frac{1}{n(n+1)}$ and we get

$$\frac{1}{(n+1)(n+2)} + \ldots + \frac{1}{(2n+1)(2n+2)} =$$

$$= \frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \ldots + \frac{1}{(2n+1)(2n+2)} - \frac{1}{n(n+1)} =$$

$$= \frac{1}{2n} + \frac{1}{2n(2n+1)} + \frac{1}{(2n+1)(2n+2)} - \frac{1}{n(n+1)} =$$

$$= \frac{2n+2}{2n(2n+1)} + \frac{n-2(2n+1)}{2n(n+1)(2n+1)} = \frac{2(n+1)^2-3n-2}{2n(n+1)(2n+1)} =$$

$$= \frac{2n^2+n}{2n(n+1)(2n+1)} = \frac{1}{2(n+1)}$$

Problem 9. 1. Prove that for every n, we have $n, (n+1)^2$ are coprime.

Proof. By the Bezout identity, it suffices to prove that 1 is a linear combination of $n, (n+1)^2$. Indeed for the integer coefficients $s = 1$ and $t = -(n+2)$ we get that

$$s(n+1)^2 + tn = 1 \cdot (n+1)^2 - (n+2) \cdot n = n^2 + 2n + 1 - n^2 - 2n = 1$$

Hence $n, (n+1)^2$ are coprime. □

2. Prove that for every n, $9^n - 2^n$ is divisible by 7.

Proof. By induction on n:

- **Base:** For $n = 0$, we get that $9^0 - 2^0 = 1 - 1 = 0$ which is divisible by 7.
- **Hypothesis**: Suppose that $9^n - 2^n$ is divisible by 7.
- **Step:** we want to prove that $9^{n+1} - 2^{n+1}$ is divisible by 7.

  $$9^{n+1} - 2^{n+1} = 9 \cdot 9^n - 2 \cdot 2^n = 7 \cdot 9^n + 2 \cdot 9^n - 2 \cdot 2^n = 7 \cdot 9^n + 2(9^n - 2^n)$$

  By the induction hypothesis $9^n - 2^n$ is divisible by 7 and thus $2(9^n - 2^n)$ is divisible by 7. Also, $7 \cdot 9^n$ is divisible by 7. Hence $9^{n+1} - 2^{n+1}$ is divisible by 7.

□

3. Prove that n is divisible by 7 if and only if n^2 is divisible by 7

   Proof. This is an "if and only if" statement do we will prove it by a doucle inclusion.

   - $\underline{\Rightarrow}$ Suppose that n we want to prove that n^2 is divisible by 7. Since n is divisible by 7, there is k such that $n = 7k$ and therefor $n^2 = (7k)^2 = 7(7k^2)$. Since $7k^2$ is an integer then n^2 is divisible by 7.
   - $\underline{\Leftarrow}$ Suppose that n^2 is divisible by 7. By the fundamental theorem of arithmetic there are (unique) q, r such that

     $$n = q7 + r, \;\; 0 \leq r < 7$$

     In order to prove that n is divisible by 7, it suffices to prove that $r = 0$. Toward a contradiction assume $r > 0$, the

     $$n^2 = (q7 + r)^2 = 47q^2 + 14rq + r^2$$

     Hence

     $$r^2 = n^2 - 49q^2 - 14qr$$

     and since $n^2, 49q^2, 14qr$ are all divisible by 7, it follows that r^2 is divisible by 7. Going over all the possibilities of $r = 1, 2, 3, 4, 5, 6$ one-by-one, we see that non of the numbers $1, 4, 9, 16, 25, 36$ is divisible by 7, contradiction r^2 being divisible by 7.

   □

4. Prove that if $\sqrt{7}$ and $\sqrt{28}$ are irrational.

Problem 10. For any function $f : \mathbb{R} \to \mathbb{R}$, denote by $Ker(f) = \{x \in \mathbb{R} \mid f(x) = 0\}$.

1. Let $f, g : \mathbb{R} \to \mathbb{R}$ be any functions. Prove that if $0 \in Ker(g)$, then $Ker(f) \subseteq ker(g \circ f)$.

Proof. Suppose that $0 \in Ker(g)$, we want to prove that $Ker(f) \subseteq Ker(g \circ f)$. By definition of $Ker(g)$, $g(0) = 0$. To prove the inclusion, let $x \in Ker(f)$, we want to prove that $x \in Ker(f \circ g)$. By definition of $Ker(f)$, $f(x) = 0$, hence by definition of composition,

$$g \circ f(x) = g(f(x)) = g(0) = 0$$

It follows by the definition of $Ker(g \circ f)$ that $x \in Ker(g \circ f)$. □

2. Give an example of such f, g such that $Ker(f) \neq Ker(g \circ f)$.
   Solution Define the functions $f, g : \mathbb{R} \to R$ by $f(x) = x$ and $g(x) = 0$, then $Ker(f) = \{0\}$ and $g \circ f(x) = g(f(x)) = g(x) = 0$, hence $Ker(g \circ f) = \mathbb{R} \neq \{0\} = Ker(f)$.

3. For any $f : \mathbb{R} \to \mathbb{R}$ and $X \subseteq \mathbb{R}$, prove that $Ker(f \restriction X) = Ker(f) \cap X$.

   Proof. We want to prove htat $Ker(f \restriction X) = Ker(f) \cap X$, which is a set equality, so we prove it by a double inclusion:

   (a) $\subseteq$ Let $x \in Ker(f \restriction x)$, then $x \in dom(f \restriction X) = X$ and $(f \restriction X)(x) = 0$, by definition of restriction, $f(x) = (f \restriction X)(x) = 0$, hence $x \in Ker(f)$. By definition of intersection $x \in Ker(f) \cap X$.

   (b) $\supseteq$ Let $x \in Ker(f) \cap X$, then $x \in Ker(f)$ and $x \in X$. It follows that $x \in Dom(f \restriction X)$ and that $(f \restriction X)(x) = f(x)$. Since $x \in Ker(f)$, $(f \restriction X)(x) = f(x) = 0$, hence $x \in Ker(f \restriction X)$.

   □

4. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a bijection then $|Ker(f)| = 1$.

   Proof. Suppose that f is a bijection and let us prove that $|Ker(f)| = 1$, namely that there is an element $x_0 \in \mathbb{R}$ such such that $Ker(f) = \{x_0\}$. Since f is a bijection, it is in particular surjective and therefore there exists x_0 such that $f(x_0) = 0$. it follows that $x_0 \in Ker(f)$. To see that $Ker(f) = \{x_0\}$, suppose toward a contradiction that this is not the case, then there is $a \in Ker(f)$ such that $a \neq x_0$. By definition of $Ker(f)$, $f(a) = 0 = f(x_0)$. Since $a \neq x_0$, this is a contradiction to f being injective. □

5. Prove or disprove, if $|Ker(f)| = 1$, then f is a bijection.

Problem 11. 1. Prove the following logical identities:

   (a) $\neg(p \Leftrightarrow p) \equiv p \Leftrightarrow \neg q$.

   (b) $(p \wedge q) \Rightarrow r \equiv \neg p \vee (q \Rightarrow r)$

   (c) $p \Rightarrow F \equiv \neg p$

   (d) $p \Rightarrow T \equiv T$.

2. Decide weather the conclusion follows from the premises:

   (a) Pre. 1: $A \Rightarrow (B \Rightarrow C)$

   (b) Pre. 2: $\neg B \vee (\neg C)$

   (c) Conclusion $\neg B \vee \neg A$.

3. Decide weather the conclusion follows from the premises:

   (a) Pre. 1: $A \wedge (\neg B \Rightarrow C)$

   (b) Pre. 2:$B \Rightarrow \neg A$

   (c) Conclusion: $\neg C \vee \neg A$.

Problem 12. Prove or disprove:

1. $\forall x, y \in \mathbb{R}. x < y \Rightarrow \exists z \in \mathbb{Q}. x < z + 1 < y$.

2. $\forall A \forall B \exists X. P(A \cap X) = P(B \cap X)$.

3. $\forall x \in \mathbb{Z}. (\exists y. 2y + 1 = x^2) \Rightarrow x + 1 \mod 3 = 0$.

Problem 13. Prove that for every $n \in \mathbb{N}_{even}$, $gcd(n, n+2) = 2$.

[Hint: Prove $gcd(n, n+2) \geq 2$ and proceed towards contradiction].

Proof. Let $n \in \mathbb{N}_{even}$, then 2 divides n. Also, $n + 2$ is even so 2 divides $n + 2$. By the definition of $gcd(n, n+2)$, it follows that $2 \leq gcd(n, n+2)$. For the other direction, $2 = (n+2) - n$ and by definition of $gcd(n, n+2)$, it divides both $n, n+2$ and thus divides 2. It follows that $gcd(n, n+2) \leq 2$. We conclude that $gcd(n, n+2) = 2$. □

Problem 14. Define for every set $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$:

$$A + r := \{a + r \mid a \in A\}$$

1. Compute $\{1, 7, -0.12\}+0.5$. No proof required. **solution** $\{1.5, 2.5, 0.38\}$
2. Let $r \in \mathbb{R}$ be any number. Compute $\mathbb{R} + r$, prove your answer.

   Proof. Prove that $\mathbb{R} + r = \mathbb{R}$ by a double inclusion. □

3. Prove the following claim:

$$\forall r \in \mathbb{R}. \mathbb{Z} + r = \mathbb{Z} \Leftrightarrow r \in \mathbb{Z}$$

4. Prove or disprove: $\forall r \in \mathbb{R}. \mathbb{N} + r = \mathbb{N} \Leftrightarrow r \in \mathbb{N}$.
   Solution Disprove, for example $r = 1 \in \mathbb{N}$ and but

$$\mathbb{N} + 1 = \{n + 1 \mid n \in \mathbb{N}\} = \mathbb{N}_+ \neq \mathbb{N}$$