Introduction to advanced mathematics- More Exercises

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Problem 1. Compute the following sets, prove your answer:

1. $\{n \in \mathbb{Z} \mid (-5) \cdot n < n\}$.
2. $\{x \in \mathbb{R} \mid \exists y \in \mathbb{R}.\exists n \in \mathbb{N}.n + y^2 = x\}$
3. $\{X \cup \{0\} \mid X \in P(\mathbb{N})\}$.
4. $\{X \in P(\mathbb{Q}) \mid X \cup \mathbb{N} \subseteq \mathbb{Z}\}$
5. $\{x \in \mathbb{R} \mid |[x, x+1] \cap \mathbb{Z}| < 2\}$

Problem 2. Prove or disprove the following statements:

1. If $A = A \setminus B$ then $B = \emptyset$.
2. If $A = A \setminus B$ then $A \cap B = \emptyset$.
3. If $A \cup B = A \cup C$ and $A \cap B = A \cap C$ then $B = C$.
4. If $A \Delta B \subseteq A \Delta B$ then $A \cap C \subseteq B$.
5. $A \cap (B \Delta C) = (A \cap B) \Delta (A \cap C)$.

Problem 3. Let A, B, C, D be sets. Prove that

$$(A \times B) \setminus (C \times D) = [(A \setminus C) \times B] \cup [A \times (B \setminus D)]$$

Problem 4. Prove the for any sets A, B:

$$A \times B = B \times A \Leftrightarrow [A = B \vee A = \emptyset \vee B = \emptyset]$$

Problem 5. Let A and B be any sets

1. $P(A \cap B) = P(A) \cap P(B)$.
2. Prove that $P(A \cup B) = P(A) \cup P(B)$ if and only if $A \subseteq B \vee B \subseteq A$.
3. $P(A \setminus B) \subseteq \{\emptyset\} \cup (P(A) \setminus P(B))$
4. If $P(A) \subseteq P(A \setminus B)$ then $A \cap B = \emptyset$.

Problem 6. Let $f : A \longrightarrow B$ and $g : B \longrightarrow C$ be function. Prove or disprove the following statements:

1. If $g \circ f$ is injective the g is injective.
2. If $g \circ f$ is injective the f is injective.
3. If $g \circ f$ is surjective then f is surjective
4. If $g \circ f$ is surjective the g is surjective.
5. If f is surjective and g is not injective then $g \circ f$ is not injective

Problem 7. Determine if the following functions are injective/surjective/ bijective. If the function is invertible, compute its inverse.

1. $f : \mathbb{N} \longrightarrow \mathbb{N}$, $f(n) = n^2 - n + 2$.
2. $f : \mathbb{R} \longrightarrow \mathbb{R}$, $f(x) = \begin{cases} 0 & x = 1 \\ \frac{1}{x-1} & x \neq 1 \end{cases}$.
3. $f : \mathbb{N} \longrightarrow P(\mathbb{N})$, $f(n) = \{k \in \mathbb{N} \mid k < n\}$.
4. $f : \mathbb{N} \times \mathbb{N} \longrightarrow P(\mathbb{N})$ $f(\langle n, m \rangle) = \{n, m\}$.
5. $f : \mathbb{N} \times \mathbb{N} \longrightarrow P(\mathbb{N})$, $f(\langle n, m \rangle) = \{n, n + m\}$
6. $f : P(\mathbb{N}) \longrightarrow P(\mathbb{N}_{even}) \times P(\mathbb{N}_{odd})$, $f(X) = \langle X \cap \mathbb{N}_{even}, X \cap \mathbb{N}_{odd} \rangle$.

Problem 8. Prove by induction the following claims:

- For every $n \geq 1$,

$$2 + 4 + 6 + \cdots + 2n = n(n + 1)$$

- For any $n \geq 1$,

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + (2n+1) \cdot 2^{2n+1} = 2 + n \cdot 2^{2n+3}$$

- For any $n \geq 1$,

$$\frac{3}{2} + \frac{9}{4} + \frac{33}{8} + \ldots + \frac{2^{2n-1}+1}{2^n} = \frac{2^{2n}-1}{2^n}$$

- For any $n \geq 1$,

$$\frac{1}{n(n+1)} + \frac{1}{(n+1)(n+2)} + \ldots + \frac{1}{(2n-1)2n} = \frac{1}{2n}$$

Problem 9.

1. Prove that for every n, we have n, $(n+1)^2$ are coprime.
2. Prove that for every n, $9^n - 2^n$ is divisible by 7.
3. Prove that n is divisible by 7 if and only if n^2 is divisible by 7
4. Prove that if $\sqrt{7}$ and $\sqrt{28}$ are irrational.

Problem 10. For any function $f : \mathbb{R} \to \mathbb{R}$, denote by $Ker(f) = \{x \in \mathbb{R} \mid f(x) = 0\}$.

1. Let $f, g : \mathbb{R} \to \mathbb{T}$ be any functions. Prove that if $0 \in Ker(g)$, then $Ker(f) \subseteq ker(g \circ f)$.
2. Give an example of such f, g such that $Ker(f) \neq Ker(g \circ f)$.
3. For any $f : \mathbb{R} \to \mathbb{R}$ and $X \subseteq \mathbb{R}$, prove that $Ker(f \restriction X) = Ker(f) \cap X$.
4. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a bijection then $|Ker(f)| = 1$.
5. Prove or disprove, if $|Ker(f)| = 1$, then f is a bijection.

Problem 11.

1. Prove the following logical identities:
   (a) $\neg(p \Leftrightarrow p) \equiv p \Leftrightarrow \neg q$.
   (b) $(p \wedge q) \Rightarrow r \equiv \neg p \vee (q \Rightarrow r)$
   (c) $p \Rightarrow F \equiv \neg p$

(d) $p \Rightarrow T \equiv T$.

2. Decide weather the conclusion follows from the premises:

   (a) Pre. 1: $A \Rightarrow (B \Rightarrow C)$
   (b) Pre. 2: $\neg B \vee (\neg C)$
   (c) Conclusion $\neg B \vee \neg A$.

3. Decide weather the conclusion follows from the premises:

   (a) Pre. 1: $A \wedge (\neg B \Rightarrow C)$
   (b) Pre. 2:$B \Rightarrow \neg A$
   (c) Conclusion: $\neg C \vee \neg A$.

Problem 12. Prove or disprove:

1. $\forall x, y \in \mathbb{R}. x < y \Rightarrow \exists z \in \mathbb{Q}. x < z + 1 < y$.

2. $\forall A \forall B \exists X. P(A \cap X) = P(B \cap X)$.

3. $\forall x \in \mathbb{Z}. (\exists y. 2y + 1 = x^2) \Rightarrow x + 1 \mod 3 = 0$.

Problem 13. Prove that for every $n \in \mathbb{N}_{even}$, $gcd(n, n+2) = 2$.

[Hint: Prove $gcd(n, n+2) \geq 2$ and proceed towards contradiction].

Problem 14. Define for every set $A \subseteq \mathbb{R}$ and $x \in \mathbb{R}$:

$$A + r := \{a + r \mid a \in A\}$$

1. Compute $\{1, 7, -0.12\} + 0.5$. No proof required.

2. Let $r \in \mathbb{R}$ be any number. Compute $\mathbb{R} + r$, prove your answer.

3. Prove the following claim:

$$\forall r \in \mathbb{R}. \mathbb{Z} + r = \mathbb{Z} \Leftrightarrow r \in \mathbb{Z}$$

4. Prove or disprove: $\forall r \in \mathbb{R}. \mathbb{N} + r = \mathbb{N} \Leftrightarrow r \in \mathbb{N}$.