ON ULTRAPOWERS AND COHESIVE ULTRAFILTERS

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Abstract. We characterize the Tukey order, the Galvin property/Cohesive ultrafilters from [30] in terms of ultrapowers. We use this characterization to measure the distance between the Tukey order and other well-known orders of ultrafilters. Secondly, we improve two theorems of Kanamori [30] from the 70's. We then study the point spectrum and the depth spectrum of an ultrafilter, and give a simple answer to Kanamori's question [30, Question 2] starting from a supercompact cardinal. Finally, we prove some consistency results regarding the depth spectrum of an ultrafilter, starting from the optimal assumption of $o(\kappa)=\kappa^{++}$, using a Woodin-like surgery argument.

1. Introduction

Among the most elegant applications of set theory, involve ultrafilter [32, 36]. For example, in Topology, ultrafilters can be used to define the Stone-Čech compactification, provide examples of topological spaces with special properties, and in the Moore-Smith convergence of nets. The latter also motivates the study of the Tukey order [41] which studies cofinal types of partially ordered sets. The Tukey order was studied extensively, both on general directed sets and on sets of the form $(\mathcal{X},\preceq)$, $(\mathcal{X},\preceq^*)$ where $\mathcal{X}$ is a filter or an ideal, and $\preceq$ is either $\supseteq$ or $\subseteq$ respectively[1]. One remarkable theorem due to Todorcevic [39] is that there are only 5 distinct cofinal types of size at most $\aleph_1$ which are provably different, but many cofinal types of cardinality $\mathfrak{c}$.

The Tukey order on ultrafilters was first considered by J. Isbell [28] in the 60's. Isbell discovered a combinatorical criterion for the maximality of ultrafilters in the Tukey order. This was later generalized to measurable cardinals in [4].

Theorem 1.1 (Isbell)**.** *Let U be an ultrafilter on λ. The following are equivalent:*

(1) $[2^\lambda]^{<\omega}\leq_T U$.

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[1] We denote by $A\subseteq^* B$ if and only if $A\setminus B$ is bounded in a cardinal which is understood from the context.

(2) $\mathbb{P} \leq_T U$ *for every directed poset* $\mathbb{P}$ *of size* 2^λ.
(3) *There is a subset* $\mathcal{A} \subseteq U$ *of cardinality* 2^λ*, such that the intersection of every infinite subset of* $\mathcal{A}$ *is not in the ultrafilter.*

Then Isbell [28] and independently Juhasz [29] constructed ultrafilters meeting the above criterion, using long independent families. There have been several constructions of such ultrafilter (see for example [40, 21, 33]).

The combinatorical criterion in (3) was independently studied in the 70's by F. Galvin [1] and A. Kanamori [30], under different names, as a weak form of regularity of ultrafilters.

Definition 1.2 (Kanamori)**.** An ultrafilter U on κ is (λ, μ)-cohesive if for every $\mathcal{A} \in [U]^\lambda$, there is $\mathcal{B} \in [\mathcal{A}]^\mu$ such that $\bigcap \mathcal{B} \in U$.

To see the translation, note that Isbell's criterion (3) translates to U not being $(2^\lambda, \omega)$-cohesive. In recent developments in Prikry-type forcing theory due to the author, Gitik, Garti and Poveda [10, 11, 12, 26, 8], cohesiveness was used under (yet) a different name- the *Galvin property*- to characterize certain intermediate models; the statement that U is (λ, μ)-cohesive is denoted in there by $\mathrm{Gal}(U, \mu, \lambda)$. These results led to the investigation of the Galvin property at the realm of measurable cardinals [7, 6, 9, 13, 23, 22].

The author and Dobrinen [4, 5] made the connection between the two parallel research streams, and developed the basic framework to study the Tukey order on measurable cardinals, generalizing results of Dobrinen and Todorcevic [19, 20] to the measurable contexts, but also discovering surprising discrepancies between the two.

Unlike other well-studied orders on ultrafilters, the Tukey order lacks an ultrapower characterization. This makes our understanding of the Tukey order somehow limited, especially when large cardinals are involved or under other axiomatic systems such as canonical inner models or under the Ultrapower Axiom (UA) [27]. This was pointed out by the author and Goldberg in [14]. The first result of this paper provides such an ultrapower characterization of the Tukey order.

Theorem 1.3. *Let* U *be an ultrafilter and* $\mathbb{P}$ *any directed set. The following are equivalent:*

(1) $\mathbb{P} \leq_T U$.
(2) *There is a thin cover*[2] $X \in M_U$ *of* $j_U''\mathbb{P}$.

We then use this characterization to measure the distance between the Tukey order and other orders on ultrafilters such as the Rudin-Keilser order, the Ketonen order. We believe that such a characterization can be useful in determining the structure of the Tukey order on σ-complete ultrafilters under UA, or at least in models in the Mitchell models of the form $L[\vec{U}]$, where

[2]See Definition 2.7

$\vec{U}$ is a coherent sequence of normal ultrafilter. We also apply our characterization to give a simple ultrapower characterization of cohesiveness/the Galvin property.

In the second part of this paper, we improve two results from [30]:

Theorem 1.4. *Suppose that U is (λ,λ)-cohesive and W is (λ,κ) cohesive for $\kappa\leq\lambda$, or vise versa. Then $U\cdot W$ is (λ,κ)-cohesive.*

Kanamori proved [31, Proposition 2.3] the special case where $\lambda=\mu=\omega_1$ and U,W are ultrafilters on ω. He also pointed out that his argument does not generalize to other cardinals. The theorem follows from the author and Dobrinen's simple formulas [4, 5] for the Tukey-type of Fubini product of κ-complete ultrafilters on a measurable cardinal κ and for certain ultrafilters on ω.

The second result we would like to improve is the following:

Theorem 1.5 ([30, Theorem 1.2(2)])**.** *Assume $2^\kappa=\kappa^+$. Any uniform ultrafilter over κ is not (κ^+,κ^+)-cohesive.*

In light of this result, the following question is natural:

Question 1.6. [Kanamori] Is it consistent that there is a measurable cardinal carrying a κ-complete ultrafilter which is (κ^+,κ^+)-cohesive?

To formulate our first theorem, we define The *character* of an ultrafilter U, as the cardinal:

$$\mathfrak{ch}(U)=\min\{|\mathcal{B}|\mid \mathcal{B}\text{ generates U}\}$$

Where $\mathcal{B}\subseteq U$ generates U (or forms a base for U) if for every $X\in U$ there is $b\in\mathcal{B}$ such that $b\subseteq X$.

Theorem 1.7. *Any uniform ultrafilter U over any cardinal κ is $(cf(\mathfrak{ch}(U)),cf(\mathfrak{ch}(U)))$-cohesive.*

Now Kanamori's theorem 1.5 is a special case of the Theorem 1.7, since if $2^\kappa=\kappa^+$, then $\mathfrak{ch}(U)=\kappa^+$.

Theorem 1.7 is optimal when $\mathfrak{ch}(U)$ is regular, as for every regular $\lambda>\mathfrak{ch}(U)$, U is $(\lambda,<\lambda)$-cohesive, and for singulars, $(\lambda,<\lambda)$-cohesive; that is, for every $\mu<\lambda$, U is (λ,μ)-cohesive. This leaves an intriguing case when $\mathfrak{ch}(U)$ is singular which we do not address in this paper. Nonetheless, note that this does not mean that U is not (λ',λ')-cohesive for $\lambda'<\mathfrak{ch}(U)$. We then investigate what is known as the *point spectrum* of an ultrafilter, denoted here by $Sp_T(U)$, which consists of all regular cardinal λ such that $\lambda\leq_T U$. This was considered before for general directed sets by Isbell [28] and earlier by Schmidt [38], and recently in connection to pcf theory by Gartside and Mamatelashvili [24], and Gilton [25]. For ultrafilters, this was indirectly addressed in [8] by Garti, Poveda and the author. We will use this to answer Question 1.6.

As it investigates cofinal types, the Tukey order is highly connected to $\mathfrak{ch}(U)$ and the *generalized ultrafilter number*, which is defined for an infinite cardinal $\kappa \geq \omega$ by

$$\mathfrak{u}_\kappa = \min\{\mathfrak{ch}(U) \mid U \text{ is a uniform ultrafilter over } \kappa\}.$$

The most studied instance is $\mathfrak{u}_\omega = \mathfrak{u}$ also known as the *ultrafilter number*. It is now a long-standing open problem whether it is consistent that $\mathfrak{u}_{\omega_1} < 2^{\omega_1}$. The main technique to separate the ultrafilter number from the continuum is to iterate Mathias forcing and create an ultrafilter with a $\subseteq^*$-decreasing generating sequence (see Definition 4.19). This technique does not generalize to higher cardinals, but some variation of it was used to show the consistency of $\mathfrak{u}_\kappa < 2^\kappa$ for measurable cardinals κ starting from a supercompact cardinal [16]. The following is completely open:

Question 1.8. Is the consistency strength of $\mathfrak{u}_\kappa < 2^\kappa$ on a measurable cardinal higher than $o(\kappa) = \kappa^{++}$?

Note that $o(\kappa) = \kappa^{++}$ is a lower bound since we must violate GCH at a measurable cardinal.

The technique of obtaining long generating sequence of ultrafilter is tightly related to the generalization of p-points considered by Kunen:

Definition 1.9 (Kunen)**.** Let U be an ultrafilter. U is called a P_λ-point if for any $\langle A_\alpha \mid \alpha < \mu\rangle \subseteq U$, where $\mu < \lambda$, there is $A \in U$ such that $A \subseteq^* A_\alpha$ for every $\alpha < \mu$.

Hence, when U is κ-complete over κ, U is a p-point precisely when it is a P_{κ^+}-point. This can of course be formulated in terms of general topological spaces by saying that a point x is a P_λ-point if every less than λ many open neighborhoods of x contain a common open neighborhood. Then an ultrafilter U on κ is a P_λ point iff it is such in the topological space $\beta\kappa \setminus \kappa$.

We use a refinement of the point spectrum, which we call the *depth Spectrum*, and define the *depth of an ultrafilter* to connect P_λ-points, Kanamori's question 1.6, and strong generating sequences in terms of consistency strength:

(1) There exists a $P_{\kappa^{++}}$-point.
(2) There is a κ-complete ultrafilter which is (κ^+, κ^+)-cohesive ultrafilter.
(3) There exists an ultrafilter on κ with a strong generating sequence of length κ^{++}.

It will be clear later that the depth spectrum is to the order $(U, \supseteq^*)$ what the decomposability spectrum of Chang and Keisler is to $(U, \supseteq)$ (i.e. to completeness).

In the last section, we prove some consistency results regarding the point and depth spectrum of a κ-complete ultrafilter over $\kappa > \omega$, starting from optimal assumptions. In particular, we provide a calculation of the point and depth spectrum in the Cohen extension. Then we prove the following:

Theorem 1.10. *Starting form $o(\kappa)=\kappa^{++}$, it is consistent that there is a normal ultrafilter U such that $St_{dp}(U)=\{\kappa^+,\kappa^{++}\}$.*

The structure of this paper is as follows:

- In Section §2: we prove our characterization of the Tukey order in terms of the ultrapower, and deduce some corollaries.
- In Section §3 we characterize cohesivness in therms of ultrapowers and improve [30, Proposition 2.3].
- In Section §4 we explore the point and depth spectrum of an ultrafilter to improve Theorem 1.5 and to address Question 1.6.
- In Section5, we present several consistency results relevant for the results of from the other sections.

Notations & global assumptions. Our notations are standard for the most part. A ultrafilter U on an infinite set X is a nonempty collection of subsets of X that is closed under intersection and superset, does not contain $\emptyset$, and for every $Y\subseteq X$, with $Y\in U$ or $X\setminus Y\in U$. We say that U is uniform if for every $Y\in U$, $|Y|=|X|$. If U is a filter on X and $f:X\to Y$ is a map, then $f_*(U)=\{B\subseteq Y\mid f^{-1}(B)\in U\}$ is also a ultrafilter, called the image ultrafilter or the pushforward ultrafilter. Many properties of U are inherited by $f_*(U)$.

For two ultrafilters U,V on X,Y respectively, we say U is Rudin-Keisler reducible to V, denoted $U\leq_{RK}V$, if there is a map $f:Y\to X$ such that $U=f_*(V)$. We call U and V Rudin-Keisler equivalent, denoted $U\equiv_{RK}V$, if there is a bijection $f:X\to Y$ such that $U=f_*(V)$. It is a standard fact that $U\leq_{RK}V$ and $V\leq_{RK}U$ imply $U\equiv_{RK}V$.

$[X]^\kappa,[X]^{<\kappa},[X]^{\leq\kappa}$ denote the sets of all subsets of X of cardinality κ, less than κ, at most κ, respectively.

Let F be a filter on X, and $f,g:X\to\kappa$. We denote by $f\leq_F g$ if $\{x\in X\mid f(x)\leq g(x)\}\in F$ and we say that f is bounded by g mod F; variations on this notation such as $f=_F g$ or $f<_F g$ should be self-explanatory. Note that if $f<_F g$ and F' is a filter extending F then $f<_{F'}g$. A function f is bounded mod F if there is $\alpha\in\kappa$ such that $f\leq_F c_\alpha$ where c_α is the constant function α. We say that f is unbounded mod F if f is not bounded mod F. Finally, our forcing convention are in Israel style, namely, $p\leq q$ means q is stronger than p.

2. A characterization of the Tukey order in terms of the ultrapower

Given two directed partially ordered sets $(\mathbb{P},\leq_{\mathbb{P}}),(\mathbb{Q},\leq_{\mathbb{Q}})$ a *Tukey map* or a *Tukey reduction* from $\mathbb{P}$ to $\mathbb{Q}$ is a function $f:\mathbb{P}\to\mathbb{Q}$ which is unbounded; that is, whenever $\mathcal{A}\subseteq\mathbb{P}$ is unbounded in $\mathbb{P}$, $f''\mathcal{A}$ is unbounded in $\mathbb{Q}$. The Tukey order, denoted by $\leq_T$, is then defined by setting $(\mathbb{P},\leq_{\mathbb{P}})\leq_T(\mathbb{Q},\leq_{\mathbb{Q}})$ iff there is a Tukey map from $\mathbb{P}$ to $\mathbb{Q}$. Schmidt found that the dual of Tukey maps are cofinal maps; A function $f:\mathbb{Q}\to\mathbb{P}$ is cofinal if for every $\mathcal{B}\subseteq\mathbb{Q}$ cofinal, $f''\mathcal{B}$ is cofinal in $\mathbb{P}$.

Proposition 2.1 (Schmidt duality [38])**.** *There is a Tukey map $f:\mathbb{P}\to\mathbb{Q}$ iff there is a cofinal map $g:\mathbb{Q}\to\mathbb{P}$*

We will mostly be interested in the Tukey order restricted to directed sets of the form $(F,\supseteq)$ where F is a filter (usually an ultrafilter) ordered by reversed inclusion. For filters we may always assume that the cofinal map is (weakly) monotone, that is, if $A\subseteq B$ then $f(A)\subseteq f(B)$. For more information regarding the Tukey order restricted to ultrafilters, we refer the reader to N. Dobrinen's survey [18].

The goal of this section is to characterize the Tukey order $\mathbb{P}\leq_T U$ for an ultrafilter over $\kappa\geq\omega$ in terms of its ultrapower, and more precisely, in terms of the existence of certain "covers" of $j_U''\mathbb{P}$. To do that we will establish a connection between these covers and functions $f:\mathbb{P}\to U$.

Definition 2.2. Let $f,g:A\to P(\kappa)$ for some set A. We say that $f=_U g$ if there is a set $Z\in U$ such that for every $a\in A$, $f(a)\cap Z=g(a)\cap Z$.

Given $X\in M_U$, we pick $Y\in V$ such that $X\subseteq j_U(Y)$. Also pick a representing function $\vec{X}=\langle X_\alpha\mid\alpha<\kappa\rangle$ (so in particular, $j_U(\vec{X})_{[id]_U}=X$), and denote by $f^{\vec{X}}_X:Y\to P(\kappa)$ the function defined by

$$f^{\vec{X}}_X(y)=\{\alpha<\kappa\mid y\in X_\alpha\}.$$

Note that if $\vec{X}'$ also represents X, then there is a set $Z\in U$ such that for every $\alpha\in Z$, $X_\alpha=X'_\alpha$. So for all $y\in Y$, $f^{\vec{X}}_X(y)\cap Z=f^{\vec{X}'}_X(y)\cap Z$, namely $f^{\vec{X}}_X=_U f^{\vec{X}'}_X$. We let f_X be some representative of this equivalence class.

In the other direction, any $f:Y\to P(\kappa)$ induces a set X_f in M_U defined by

$$M_U\models X_f=\{y\in j_U(Y)\mid [id]_U\in j_U(f)(y)\}.$$

Once again, note that if $f=_U g$, then $X_f=X_g$.

Proposition 2.3. *For any $X\in M_U$, any choice of Y so that $X\subseteq j_U(Y)$, $X_{f_X}=X$. Also for any $f:Y\to P(\kappa)$, $f_{X_f}=_U f$.*

Proof. Note that $X_{f_X}=\{q\in j_U(Y)\mid [id]_U\in j_U(f_X)(q)\}$, then for every $q\in j_U(Y)$,

$$\begin{aligned} q\in X_{f_X} &\text{ iff } [id]_U\in j_U(f_X)(q)\\ &\text{ iff } [id]_U\in\{\alpha<j_U(\kappa)\mid q\in j_U(\vec{X})_\alpha\}\\ &\text{ iff } q\in j_U(\vec{X})_{[id]_U}\\ &\text{ iff } q\in X\end{aligned}$$

For the second part, X_f is represented by $\vec{X}=\langle X_\alpha\mid\alpha<\kappa\rangle$, where $X_\alpha:=\{p\in Y\mid\alpha\in f(p)\}$. Let $p\in Y$, then

$$f^{\vec{X}}_{X_f}(p)=\{\alpha<\kappa\mid p\in(X_f)_\alpha\}=\{\alpha<\kappa\mid\alpha\in f(p)\}=f(p)$$

Hence $f = f_X^{\vec{X}} =_U f_X$

□

Definition 2.4. Let $Z \subseteq M_U$. We say that $X \in M_U$ *covers* Z if for every $p \in Z$, $M_U \models p \in X$.

If M_U is well-founded (and therefore identified with its transitive collapse), then a cover is just a superset. From now on, we will write $x \in y$ for elements in M_U where we actually mean that $M_U \models x \in y$. Similarly, $A \cap B$ for $A, B \in M_U$ is defined in M_U as the set of all p such that $M_U \models p \in A \wedge p \in B$, and so on.

Claim 2.5. *For any function $f : Y \to P(\kappa)$ and $Z \subseteq Y$, X_f covers $j_U'' Z$ iff $f \restriction Z : Z \to U$.*

Proof. $\underline{\Longrightarrow:}$ follows easily from Loś Theorem.

$\underline{\Longleftarrow:}$ Let $p \in Z$, then $j_U(f)(j_U(p)) = j_U(f(p))$. Since $f(p) \in U$, $[id]_U \in j_U(f(p))$, and by the definition of X_f, $j_U(p) \in X_f$.

□

By the previous claim, we conclude that

Corollary 2.6. *For any set $X \in M_U$, and any sets $Z \subseteq Y$, X covers $j_U'' Z$ iff $f_X \restriction Z : Z \to U$.*

Proof. Since $X = X_{f_X}$, the corollary follows by applying the claim to f_X. □

Now we translate between properties of f and properties of X. The first, is unboundedness:

Definition 2.7. Let U be an ultrafilter, and $\mathbb{P}$ a directed set. We say that a set $X \in M_U$ is a *thin cover* of $\mathbb{P}$ if $j_U'' P \subseteq X$ and for any unbounded set $\mathcal{A} \subseteq \mathbb{P}$, $j_U(\mathcal{A}) \not\subseteq \mathbb{P}$.

Lemma 2.8. *Let $\mathbb{P}$ be a directed set and $f : Y \to P(\kappa)$ such that $\mathbb{P} \subseteq Y$. Then $f \restriction \mathbb{P} : \mathbb{P} \to U$ is unbounded iff X_f is a thin cover.*

Proof. Suppose $f \restriction \mathbb{P} \to U$ is unbounded. Then by Corollary 2.6, X_f is indeed a cover. To see it is thin, suppose $j_U(\mathcal{A}) \subseteq X_f$, then

$$[id]_U \in \bigcap_{B \in j_U(\mathcal{A})} j_U(f)(B) = j_U(\bigcap_{B \in \mathcal{A}} f(B)).$$

Hence $\bigcap_{B \in \mathcal{A}} f(B) \in U$, which means that $f'' \mathcal{A}$ is bounded in U. Since $f \restriction \mathbb{P}$ is unbounded, $\mathcal{A}$ must have been bounded. In the other direction, suppose that X_f is a thin cover. Then by Corollary 2.6, $f \restriction \mathbb{P} : \mathbb{P} \to U$. To see that $f \restriction \mathbb{P}$ is unbounded, let $\mathcal{A} \subseteq \mathbb{P}$ be unbounded, since $j_U(\mathcal{A}) \not\subseteq X_f$,

$$[id]_U \notin \bigcap_{B \in j_U(\mathcal{A})} j_U(f)(B) = j_U(\bigcap_{B \in \mathcal{A}} f(B)).$$

Hence $f'' \mathcal{A}$ is unbounded in U. □

Corollary 2.9. *For any cover $X \in M_U$, X is a thin cover iff $f_X \restriction \mathbb{P}$ is unbounded.*

This gives a characterization of the Tukey order in terms of covers:

Theorem 2.10. *Let U be an ultrafilter and $\mathbb{P}$ any directed set. The following are equivalent:*

(1) $\mathbb{P} \leq_T U$.
(2) *There is a thin cover $X \in M_U$ of $j_U''\mathbb{P}$.*

Corollary 2.11. *Let U, W be ultrafilters. The following are equivalent:*

(1) $W \leq_T U$.
(2) *There is a cover $X \in M_U$ of $j_U''W$ such that if $\bigcap \mathcal{A} \notin W$, then $j_U(\mathcal{A}) \not\subseteq X$.*

Remark 2.12. We are crucially missing a characterization of a cofinal map $g : U \to W$ in terms of the ultrapower by W.

When we cover $j_U''W$, it is tempting to require that the cover $X \in M_U$ is a filter. However, as we will further notice, this corresponds to Tukey maps with additional properties. From now on, we only consider $\mathbb{P} = W$ for some ultrafilter W and our canonical choice of Y would be $P(\kappa)$. So we consider functions $f : P(\kappa) \to P(\kappa)$.

Definition 2.13. We say that a function $f : P(\kappa) \to P(\kappa)$ is:

(1) monotone, if $A \subseteq B \Rightarrow f(A) \subseteq f(B)$
(2) semi-additive, if $f(A) \cap f(B) \subseteq f(A \cap B)$.
(3) additive if $f(A) \cap f(B) = f(A \cap B)$.
(4) μ-semi-additive if for any $\langle A_i \mid i < \lambda\rangle \in [P(\kappa)]^{<\mu}$, $\bigcap_{i<\mu} f(A_i) \subseteq f(\bigcap_{i<\mu} A_i)$
(5) μ-additive if for any $\langle A_i \mid i < \lambda\rangle \in [P(\kappa)]^{<\mu}$, $\bigcap_{i<\mu} f(A_i) = f(\bigcap_{i<\mu} A_i)$.
(6) negative if $f(\kappa \setminus A) = \kappa \setminus f(A)$.
(7) an homomorphism if f is negative and additive.

It is not hard to check that f is μ-additive iff it is μ-semi-additive and monotone. We say that a set $X \subseteq P(\kappa)$ is ultra, if for every A,

$$A \in X \text{ xor } \kappa \setminus A \in X.$$

The proof of the following propositions is simple.

Proposition 2.14. (1) *f is monotone $\Rightarrow X_f$ is upwards closed.*
(2) *f is semi-additive $\Rightarrow X_f$ is closed under intersection.*
(3) *f is additive $\Rightarrow X_f$ is a filter.*
(4) *f is μ-additive $\Rightarrow X_f$ is a μ-complete filter.*
(5) *f is negative $\Rightarrow X_f$ is ultra.*
(6) *f is an homomorphism $\Rightarrow X_f$ is an ultrafilter.*

□

Proposition 2.15.
(1) *X is upwards closed$\Rightarrow$ then there is a monotone f such that $f_X = f$.*
(2) *X is closed under intersection $\Rightarrow$ there is a semi-additive f so that $f_X =_U f$ is*
(3) *X is a filter $\Rightarrow$ there is an additive f so that $f_X =_U f$.*
(4) *X is a μ-complete filter $\Rightarrow$ there is a μ-additive f so that $f_X =_U f$.*
(5) *X is ultra $\Rightarrow$ there is a negative f so that $f_X =_U f$.*
(6) *X is an ultrafilter $\Rightarrow$ there is an homomorphism f such that $f_X =_U f$.*

□

Corollary 2.16.
(1) *There is a thin upward closed cover iff there is a monotone Tukey map.*
(2) *There is a thin cover closed under intersection iff there is a semi-additive Tukey map.*
(3) *There is a thin filter cover iff there is an additive Tukey map.*
(4) *There is a thin μ-complete filter cover iff there is a μ-additive Tukey map.*
(5) *There is an ultra thin cover iff there is a negative Tukey map.*
(6) *There is an ultrafilter thin cover iff there is an homomorphism Tukey map.*

□

By the ultrafilter lemma we get that:

Corollary 2.17. *There is an additive Tukey map iff there is an homomorphism Tukey map.*

Lemma 2.18. *Any cover $X \in M_U$, closed under intersections, such that $j_U(\mathcal{C}) \nsubseteq X$ for every $\mathcal{C} \subseteq W$ with $\bigcap C = \emptyset$ must be a thin cover.*

Proof. It remains to see that X does not contain the image of a set $\mathcal{C}$ such that $\bigcap \mathcal{C} \notin W$. Suppose otherwise, then $j_U(\mathcal{C}) \subseteq X$. Let

$$\mathcal{C}' = \{Y \cap \overline{\bigcap \mathcal{C}} \mid Y \in \mathcal{C}\}.$$

Then $\bigcap \mathcal{C}' = \emptyset$. But since X covers $j''_U W$, and $\overline{\bigcap \mathcal{C}} \in W$, $j_U(\overline{\bigcap \mathcal{C}}) \in X$. Also since X is closed under intersection and $j_U(\mathcal{C}) \subseteq X$, then $j_U(\mathcal{C}') = \{Y \cap j_U(\overline{\bigcap \mathcal{C}}) \mid Y \in j_U(\mathcal{C})\} \subseteq X$, contradicting the assumption regarding X. □

Next, we would like to use the above characterization to measure the distance between the Tukey order and other orders on ultrafilters. Recall that the Rudin-Keisler order is defined as follows:

Definition 2.19. Let U, W be ultrafilters over κ respectively. We say that $W \leq_{RK} U$ if there is $f : \kappa \to \kappa$ such that

$$W = f_*(U) = \{A \subseteq \kappa \mid f^{-1}[A] \in U\}.$$

Given a Rudin-Keisler projection $f : \kappa \to \kappa$, we can define $F_f : P(\kappa) \to P(\kappa)$ by $F_f(A) = f^{-1}[A]$. Then we have the following properties:

(1) F_f is an ∞-additive homomorphism. That is, for any $\mathcal{A} \subseteq P(\kappa)$, $F_f(\bigcap \mathcal{A}) = \bigcap F''_f \mathcal{A}$.
(2) $F_f \restriction W : W \to U$ is Tukey.

Corollary 2.20. *$W \leq_{RK} U$ then $W \leq_T U$*

Observe that the Rudin-Keisler order is characterized by:

$$W \leq_{RK} U \text{ iff } \bigcap j''_U W \neq \emptyset.$$

Given any $\alpha \in \bigcap j''_U W$ we can explicitly define the cover

$$X = p_\alpha^{j_U(\kappa)} := \{A \subseteq j_U(\kappa) \mid \alpha \in A\}.$$

A principal ultrafilter is always ∞-complete this ultrafilter cover of $j''_U W$. This simple observation enables us to characterize the Rudin-Keisler order in terms of unbounded maps:

Corollary 2.21. *$W \leq_{RK} U$ iff there is an ∞-additive map f such that $f \restriction W$ is a Tukey reduction.*

Proof. From left to right follows from the previous paragraph. In the other direction, if f is such a map, then there is a thin filter cover X of $j''_U W$ which is ∞-complete. This means that $\bigcap X \in X$. Since X is thin, X is a proper filter and thus $\emptyset \notin X$. It follows that $\bigcap X \neq \emptyset$ and in particular $\bigcap j''_U W \neq \emptyset$. It follows that $U \leq_{RK} W$. □

Remark 2.22. Ther is no hope of characterazing the Tukey order on all ultrafilters using function $f : \kappa \to \kappa$, as it is consistent that there are 2^{2^κ}-many incomperable ultrafilters in the Tukey order, and since there is always a Tukey-top (i.e. maximal) ultrafilter.

Next we address continuity of functions $f : P(\kappa) \to P(\kappa)$.

Definition 2.23. A function $f : P(\kappa) \to P(\kappa)$ is continuous if for every $\alpha < \kappa$ there is $\xi_\alpha < \kappa$ such that for every $A \in W$, $f(A) \cap \alpha$ depends only on $A \cap \xi_\alpha$. f is Lipschitz if we can pick $\xi_\alpha = \alpha$ and super-Lipschitz if $f(A) \cap \alpha + 1$ depends only on $A \cap \alpha$.

Definition 2.24. We say that X concentrates on A if for every B, C with $B \cap A = C \cap A$, $B \in X$ iff $C \in X$.

Remark 2.25. If X is upwards closed, and X concentrates on $A \subseteq j_U(\kappa)$, then $A \in X$: Indeed $j_U(\kappa) \in X$ and $A \cap A = A = A \cap j_U(\kappa)$, hence $A \in X$.

If X is moreover a filter then X concentrates on any $A \in X$: Let $B \in X$ and suppose that $B \cap A = C \cap A$, then $B \cap X \in A$ and therefore $C \cap X \in A$ and therefore $C \in A$.

Proposition 2.26. *Let U be an ultrafilter. If $f : P(\kappa) \to P(\kappa)$ is continuous with parameters ξ_α, then in M_U, X_f concentrates on $[\alpha \mapsto \xi_{\alpha+1}]_U$. In particular, if f is Lipschitz then X_f concentrates on $[id]_U + 1$ and if f is super-Lipschitz then X_f concentrates on $[id]_U$.*

Proof. Suppose that $X \in A$ and $Y \subseteq j_U(W)$ is such that $Y \cap \xi_{[id]_U+1} = X \cap \xi_{[id]_U+1}$, then $j_U(f)(Y) \cap [id]_U + 1 = j_U(f)(X) \cap [id]_U + 1$ and therefore $Y \in A$. □

Proposition 2.27. *If $X \in M_U$ concentrates on $\delta < j_U(\kappa)$ then there is a continuous function $f : P(\kappa) \to P(\kappa)$ such that $f_X =_U f$. In particular, if X concentrates on $[id]_U+1$, then f above cab be Lipschitz and if it concentrates on $[id]_U$ then super-Lipschitz.*

Proof. Suppose that X concentrates on $\delta = [\alpha \mapsto \delta_\alpha]_U$, and let

$$Z = \{\alpha < \kappa \mid X_\alpha \text{ concetrates on } \delta_\alpha\} \in U$$

Let $\vec{X}$ be a representing sequence for X so that for every α, X_α concentrates on δ_α, and let $f = f^{\vec{X}}_X$. By the definition of $f^{\vec{X}}_X$, $f(A) = \{\alpha < \kappa \mid A \in X_\alpha\}$. Hence f is continuous with parameters δ_α. Indeed, for every $\beta < \kappa$, if $X \cap \delta_\beta = Y \cap \delta_\beta$, then for every $\gamma \leq \beta$ $X \cap \delta_\gamma = Y \cap \delta_\gamma$ and since A_γ concentrates on δ_γ, we have that $\gamma \in f(X)$ iff $\gamma \in f(Y)$. Hence $f(X) \cap \beta + 1 = f(Y) \cap \beta + 1$. Finaly $f = f^{\vec{X}}_X =_U f_X$. □

The following theorem was proven in [4]. We will need to derive some fine corollaries from the proof, so let us reproduce the proof here:

Theorem 2.28. *If U is a p-point ultrafilter, and $f : U \to W$ is monotone, then there is $X^* \in U$ such that $f \restriction (U \restriction X^*)$ is continuous.*

Proof. Suppose that U is any p-point κ-complete ultrafilter, W is any ultrafilter on κ, and $f : U \to W$ is monotone. Let $\pi : \kappa \to \kappa$ represent κ in M_U. Then by p-pointness we may assume that π is almost one-to-one. Denote by $\rho_\alpha = \sup(\pi^{-1}[\alpha+1])$ and fix any sequence $\gamma_\alpha < \kappa$. For $\alpha < \kappa$. let us define a sequence of sets X_α: For every $s \subseteq \rho_\alpha$ and $\delta < \gamma_\alpha$, if there is a set $Y \in U$ such that $Y \cap \rho_\alpha = s$ and $\delta \notin f(Y)$, pick $Y_{s,\delta} = Y$. Otherwise let $Y_{s,\delta} = \kappa$. Define

$$X_\alpha = \bigcap_{s,\delta} Y_{s,\delta} \in U$$

Claim 2.29. *X_α has the property that for any $Y \in U$, if $Y \setminus \rho_\alpha \subseteq X_\alpha \setminus \rho_\alpha$, $f(Y) \cap \gamma_\alpha = f((Y \cap \rho_\alpha) \cup (X_\alpha \setminus \rho_\alpha)) \cap \gamma_\alpha$.*

Proof. Indeed, $Y \subseteq (Y \cap \rho_\alpha) \cup (X_\alpha \setminus \rho_\alpha)$, so by monotonicity, we get " $\subseteq$ ". In the other direction, if $\delta \notin f(Y) \cap \gamma\alpha$, let $s = Y \cap \rho_\alpha$, we have that $s \cup X_\alpha \setminus \rho_\alpha \subseteq Y_{s,\delta}$ and since $\delta \notin f(Y_{s,\delta})$, it follows again by monotonicity that $\delta \notin f(s \cup X_\alpha \setminus \rho_\alpha)$. □

By p-pointness, let $X^* = \Delta^*_{\alpha<\kappa} X_\alpha$ be the modified diagonal intersection defined as

$$\Delta^*_{\alpha<\kappa} X_\alpha = \{\nu < \kappa \mid \forall \alpha < \pi(\nu),\ \nu \in X_\alpha\} \in U.$$

Then for every $\alpha < \kappa$ we have $X^* \setminus \rho_\alpha \subseteq X_\alpha$. Now for every $\alpha < \kappa$ and every $Y \subseteq X^*$ we have that $f(Y) \cap \gamma_\alpha = f((Y \cap \rho_\alpha) \cup (X^* \setminus \rho_\alpha)) \cap \gamma_\alpha$. Hence $f \restriction U \restriction X^*$ is continuous. □

Suppose that U is normal, in particular $\rho_\alpha = \alpha + 1$ and suppose that $\gamma_\alpha = \alpha+2$. Denote by $2^{<\kappa}$ the set of all binary functions f with $\mathrm{dom}(f) \in \kappa$. Define $\hat{f}(h)$ for $h \in 2^\alpha$. For $\alpha = 0$, $\hat{f}(h) = \emptyset$. For $\alpha + 1$ successor, define

$$\hat{f}(h) = f((\chi^{-1}(h) \cap X^*) \cup X^* \setminus \mathrm{dom}(h)) \cap \alpha + 2,$$

where $\chi : P(\kappa) \to 2^\kappa$ is the map sending X to its indicator function $\chi(X)$. For limit α, define $\hat{f}(h) = \bigcup_{\beta<\alpha} \hat{f}(h \restriction \beta + 1)$. This defined $\hat{f} : 2^{<\kappa} \to P(\kappa)$. Now define $f^* : P(\kappa) \to P(\kappa)$ by $f^*(X) = \bigcup_{\alpha<\kappa} \hat{f}(\chi(X) \restriction \alpha)$.

Proposition 2.30. *Suppose that U is normal, then*

1. *if $h_1 \subseteq h_2$, then $\hat{f}(h_1) \sqsubseteq \hat{f}(h_2)$.*
2. *f^* is Lipschitz continuous.*
3. *$f \restriction (U \restriction X^*) \subseteq f^*$.*
4. *f^* is monotone.*
5. *If f is cofinal then f^* is cofinal.*
6. *If f is unbounded then f^* is unbounded.*
7. *If f is cofinal and $U \neq W$, then f^* is super-Lipschits.*

Proof. (1) We may assume that $\mathrm{dom}(h_1), \mathrm{dom}(h_2)$ are successor cardinals as the limit case follows from the successor case and the definition of $\hat{f}$. Suppose that $h_1 \subseteq h_2$ have successor domains $\alpha_1 \leq \alpha_2$, then $\chi(h_2) \cap X^* \cap \alpha_1 = \chi(h_1) \cap X^*$, we get that $\hat{f}(h_1) = f((\chi(h_1) \cap X^*) \cup X^* \setminus \alpha_1) \cap \alpha_1 + 1$. Also, taking $Y = (\chi(h_2) \cap X^*) \cup X^* \setminus \alpha_2$, we have that $Y \subseteq X^*$. By the property of f, we have that $f(Y) \cap \alpha_1 + 1 = f((Y \cap \alpha_1) \cup X^* \setminus \alpha_1) \cap \alpha_1 + 1$. Hence

$$\begin{aligned}
\hat{f}(h_2) \cap \alpha_1 + 1 &= f((\chi(h_2) \cap X^*) \cup X^* \setminus \alpha_2) \cap \alpha_1 + 1 \\
&= f((Y \cap \alpha_1) \cup X^* \setminus \alpha_1) \cap \alpha_1 + 1 \\
&= f((\chi(h_2) \cap X^* \cap \alpha_1) \cup (X^* \setminus \alpha_1)) \cap \alpha_1 + 1 \\
&= f((\chi(h_1) \cap X^*) \cup (X^* \setminus \alpha_1)) \cap \alpha_1 + 1 = \hat{f}(h_1).
\end{aligned}$$

Thus $\hat{f}(h_1) \sqsubseteq \hat{f}(h_2)$.

(2) follows from (1). Indeed, since $f^*(X)$ is the $\sqsubseteq$-increasing union of $\hat{f}(\chi(X) \restriction \alpha)$, we see that

$$f^*(X) \cap \alpha = \hat{f}(\chi(X) \restriction \alpha) \cap \alpha,$$

and if $X \cap \alpha = Y \cap \alpha$ then $\chi(X) \restriction \alpha = \chi(Y) \restriction \alpha$.

(3) If $Y \subseteq X^*$ and $Y \in U$, then for every $\alpha < \kappa$, $Y \cap X^* \cap \alpha+1 = Y \cap \alpha+1$ and

$$f(Y)\cap\alpha+1 = f((Y\cap\alpha+1)\cup(X^*\setminus\alpha+1))\cap\alpha+1 = \hat{f}(\chi(Y)\restriction\alpha+1) = f^*(Y)\cap\alpha+1.$$

Since this is true for every α, $f^*(Y) = f(Y)$.

(4) For every $X \subseteq \kappa$ and for every $\alpha < \kappa$,

$$\hat{f}(\chi(X)\restriction\alpha+1) = f(X\cap X^*\cap\alpha+1)\cup X^*\setminus\alpha+1)\cap\alpha+1.$$

Since f is monotone, we conclude that if $X_1 \subseteq X_2$, then for every α, $f^*(X_1)\cap \alpha+1 \subseteq f^*(X_2)\cap\alpha+1$. resulting in $f^*(X_1) \subseteq f^*(X_2)$.

(5) If f is cofinal, then $f''U \restriction X^*$ is also cofinal (since $U\restriction X^*$ is cofinal in U). By (3), $f^{*''}U\restriction X^*$ is cofinal and therefore f^* has a cofinal image. This together with (4) suffices for f^* to be cofinal.

(6) Suppose that f is unbounded. To see that f^* is unbounded, suppose that $\mathcal{A} \subseteq U$ is unbounded, then also $\mathcal{A}\restriction X^* = \{A\cap X^* \mid X \in \mathcal{A}\}$ is unbounded. Since f is unbounded then $f''\mathcal{A}\restriction X^*$ in unbounded. Now it is not hard to see that for every $Y \subseteq \kappa$, $f^*(Y) = f^*(Y\cap X^*) = f(Y\cap X^*)$ and therefore $f^{*''}\mathcal{A}$ is also unbounded.

(7) First note that by the proof of (2), for α successor we get even more, that $f^*(X)\cap\alpha+1 = \hat{f}(\chi(X)\restriction\alpha)$, and therefore if $X\cap\alpha = Y\cap\alpha$ then $f^*(X)\cap\alpha+1 = f^*(X)\cap\alpha+1$. Now let $X_0 \in U$ be such that $X_0^c \in W$. Since f is cofinal, there is Y such that $f(Y) \subseteq X_0^c$ and we may take our X^* so that $X^* \subseteq X_0\cap Y$ and in particular $f(X^*) \subseteq X_0^c$. Note that for every h, $\hat{f}(h) \subseteq f(X^*)$ and therefore for every X, $f^*(X) \subseteq f(X^*)$. Let $\alpha < \kappa$ and suppose that X, Y are such that $X\cap\alpha = Y\cap\alpha$. By (2), $f^*(X)\cap\alpha = f^*(Y)\cap\alpha$. Let us split into cases. If $\alpha \in X_0$, then $\alpha \notin f^*(X)$ and $\alpha \notin f^*(Y)$. Hence $f^*(X)\cap\alpha+1 = f^*(Y)\cap\alpha+1$. If $\alpha \notin X_0$, then $\alpha \notin X^*$, hence $\alpha \notin X\cap X^*$ and $\alpha \notin Y\cap X^*$. It follows that

$$X\cap X^*\cap\alpha+1 = X\cap X^*\cap\alpha = Y\cap X^*\cap\alpha = Y\cap X^*\cap\alpha+1$$

We conclude that

$$f^*(X)\cap\alpha+1 = f^*(X\cap X^*)\cap\alpha+1 = f^*(Y\cap X^*)\cap\alpha+1 = f^*(Y)\cap\alpha+1.$$

□

Corollary 2.31. *If $W \leq_T U$, and U is normal and $W \neq U$, then there is an upwards closed cover $X \in M_W$ of $j''_U W$ such that X concentrates on $[id]_U$.*

The above is closely related to the Ketonen order. The reason is that the Ketonen ordered is defined by $U <_{\mathbb{K}} W$ if there is a cover X such that X is an ultrafilter concentrating on $[id]_U$.

Since we lack the characterization of cofinal maps in the ultrapower, there are some properties of the cover X we are missing. These properties might be useful in an attempt to prove that normal ultrafilters are Tukey minimal.

Question 2.32. Is there a property of the cover X_f that is equivalent to f being cofinal? How about monotone continuous and cofinal?

3. On Cohesive ultrafilters

Recall that an ultrafilter U is (λ,μ)-cohesive (or alternatively, $\text{Gal}(U,\mu,\lambda)$ holds) if for any $\mathcal{A}\in[U]^\lambda$ there is $\mathcal{B}\in[\mathcal{A}]^\mu$ such that $\bigcap\mathcal{B}\in U$. The results of the previous section provide an elegant characterization of the cohesiveness in terms of the ultrapower:

Theorem 3.1. *Let U be an ultrafilter over $\kappa\geq\omega$, and $\mu\leq\lambda$ be any cardinals. Then the following are equivalent:*

1. *U is (λ,μ)-cohesive.*
2. *Any cover $X\in M_U$ of $j_U''\lambda\subseteq X$ contains a set of the form $j_U(Y)$ for some $Y\in[\lambda]^\mu$.*

Proof. Assume that U is (λ,μ)-cohesive and let $X=[\beta\mapsto X_\beta]_U$ be any cover as in (2). For every $i<\lambda$ let $A_i=\{\beta<\kappa\mid i\in X_\beta\}$, then by Łoś Theorem, $A_i\in U$. By (λ,μ)-cohesiveness, there is $Y\in[\lambda]^\mu$ such that $A^*=\bigcap_{i\in Y}A_i\in U$. It follows that for every $\beta\in A^*$, $Y\subseteq X_\beta$ and therefore (again by Łoś) $j_U(Y)\subseteq X$.

In the other direction, suppose that (2) holds and let us prove that U is (λ,μ)-cohesive. Let $\langle A_i\mid i<\lambda\rangle\subseteq U$. Working in M_U, consider

$$j_U(\langle A_i\mid i<\lambda\rangle)=\langle A'_i\mid i<j_U(\lambda)\rangle$$

and define $X=\{\beta<j_U(\lambda)\mid [id]_U\in A'_\beta\}\in M_U$. Then, since $A'_{j_U(\beta)}=j_U(A_\beta)$, $[id]_U\in A'_{j_U(\beta)}$ and $j_U(\beta)\in X$. It follows that X covers $j_U''\lambda$. By (2) there is $Y\in[\lambda]^\mu$ such that $j_U(Y)\subseteq X$ and therefore $\bigcap_{i\in Y}A_i\in U$. To see this, note that

$$j_U(\bigcap_{i\in Y}A_i)=\bigcap_{i\in j_U(Y)}A'_i$$

and since $j_U(Y)\subseteq X$, $[id]_U\in A'_i$ for all $i\in j_U(Y)$ hence $[id]_U\in j_U(\bigcap_{i\in Y}A_i)$ □

Remark 3.2.
1. Requiring that the cover X contains a set of the form $j_U(Y)$ where Y has size μ is equivalent to representing $X=[\beta\mapsto X_\beta]_U$ and requiring the existence of $A\in U$ such that $\bigcap_{\beta\in A}X_\beta$ has size μ.
2. Condition (2) above can be replaced with the following: Any $X\in M_U$ such that (in V) $|X\cap j_U''\lambda|=\lambda$ contains a set of the form $j_U(Y)$ for some $Y\in[\lambda]^\mu$. The reason is that the existence of a bijection $\varphi:\lambda\to j_U^{-1}[X]$ enables to convert the set X to a full cover of $j_U''\lambda$.

Let us now reproduce the characterization of Tukey-top ultrafilters using Theorem 3.1 and Theorem 2.10

Corollary 3.3. *Let U be an ultrafilter over $\kappa\geq\omega$, and $\mu\leq\lambda$ are such that $\lambda^{<\mu}=\lambda$. The following are equivalent:*

1. *For any μ-directed set $\mathbb{P}$, such that $|\mathbb{P}|\leq\lambda$, $\mathbb{P}\leq_T(U,\supseteq)$.*
2. *$([\lambda]^{<\mu},\subseteq)\leq_T(U,\supseteq)$.*

(3) *U is not (λ,μ)-cohesive.*

Proof. Clearly, (1) implies (2). To see that (2) implies (3), by Theorem 2.10 there is a thin cover X of $j''_U[\lambda]^{<\mu}$. Define $X' = \{Y \mid \{Y\} \in X\} \in M_U$ Then $j''_U\lambda \subseteq X'$. Note that if $I \in [\lambda]^\mu$, then $I^* = \{\{i\} \mid i \in I\}$ is unbounded in $[\lambda]^{<\mu}$ and therefore there $j_U(I^*) \not\subseteq X$. therefore, there is $i \in j_U(I)$ such that $\{i\} \notin X$, namely, $i \notin X'$. So X' is a thin cover of $j''_U\lambda$ and by Theorem 3.1, U is not (λ,μ)-cohesive. To see that (3) implies (1), Let $\mathbb{P}$ be any μ-directed set of size $\leq \lambda$ and let $f : \mathbb{P} \to \lambda$ be injective. By Theorem 3.1 find a thin cover X of $j''_U\lambda$. Then $X' = j_U^{-1}(f)[X]$ is a thin cover of $j''_U\mathbb{P}$. □

By Galvin's Theorem [1], every normal (or even p-point) ultrafilter is (κ^+,κ)-cohesive. Hence we get the following:

Corollary 3.4. *If U is normal (or even a p-point), then whenever $X \in M_U$ covers $j''_U\kappa^+$, there is $Y \in [\kappa^+]^\kappa$ such that $j_U(Y) \subseteq X$.*

An improvement of Galvin's theorem can be found in [2] to iterated sums of p-points. So the above corollary holds in this generality as well.

Kanamori proved that if $\{U\} \cup \{W_n \mid n < \omega\rangle\}$ is a set of (ω_1,ω_1)-cohesive ultrafilters on ω, then $\sum_U W_n$ is (ω_1,ω)-cohesive. He then says that this does not generalize to $\kappa > \omega$. We will prove that to some extent his result do generalize to $\kappa > \omega$, and that on measurable cardinals we can even say a bit more. We say that a directed set $\mathbb{P}$ is (λ,μ)-cohesive, if for every $\langle p_\alpha \mid \alpha < \lambda\rangle \subseteq \mathbb{P}$ there is $I \in [\lambda]^\mu$ such that $\{p_i \mid i \in I\}$ is bounded. It is not hard to see that cohesiveness is an invariant of the Tukey order:

Fact 3.5. If $\mathbb{P} \leq_T \mathbb{Q}$ and $\mathbb{Q}$ is (λ,μ)-cohesive, then $\mathbb{P}$ is (λ,μ)-cohesive. So if $\mathbb{P} \equiv_T \mathbb{Q}$, then $\mathbb{Q}$ is (λ,μ)-cohesive if and only if $\mathbb{P}$ is (λ,μ)-cohesive.

In fact, similar to ultrafilters, it is possible to show that if $\lambda^{<\mu} = \lambda$, $\mathbb{P}$ is not (λ,μ)-cohesive exactly when $([\lambda]^{<\mu},\subseteq) \leq_T \mathbb{P}$. We will also need the following theorem.

Theorem 3.6 (B.-Dobrinen [4])**.** *Let U,W be κ-complete ultrafilters over $\kappa > \omega$. Then $U \cdot W \equiv_T U \times W$*

Theorem 3.7. *Let U,W be κ-complete ultrafilters over $\kappa > \omega$. Suppose that U is (λ,λ)-cohesive and W is (λ,μ)-cohesive (or the other way around). Then $U \cdot W$ is (λ,μ)-cohesive.*

Proof. First, by Theorem 3.6, $U \cdot W \equiv_T U \times W$. By the previous fact, it suffices to prove that $U \times W$ is (λ,κ)-cohesive. Given $\langle (A_\alpha, B_\alpha) \mid \alpha < \lambda\rangle$ we need to find μ-many of the pairs which have a lower bound. Indeed since U is (λ,λ)-cohesive, there is $I \subseteq \lambda$ of size λ such that $B^* = \bigcap_{i\in I} B_i \in U$. Applying (λ,μ)-cohesiveness to $\langle A_i \mid i \in I\rangle$, find $J \subseteq I$ of size μ such that $A^* = \bigcap_{j\in J} A_j \in W$. Then $\langle\langle A_j, B_j\rangle \mid j \in J\rangle$ is bounded by (A^*, B^*). □

On ω, it is not true that for every two ultrafilter U,W, $U \cdot W \equiv U \times W$. Indeed, Dobrinen-Todorcevic [19] gave an exmaple (under $\mathfrak{u} < \mathfrak{d}$) of an ultrafilter (even a p-point) U which is not Tukey equivalent to its Fubini square. However, by [5, Cor. 1.9], if $W\cdot W\equiv_T W$, then $U\cdot W\equiv_T U\times W$ and the above proof works.

Corollary 3.8. *Fix any two ultrafilters U,W over ω such $W\cdot W\equiv_T W$. If U is (λ,λ)-cohesive and W is (λ,μ)-cohesive then $U\cdot W$ is (λ,μ)-cohesive.*

In [5], the class of ultrafilters which satisfies $W\cdot W\equiv_T W$ was investigated, and includes rapid p-points (and almost rapid p-points- see [3]), many instances of generic ultrafilters for $P(\omega)/I$, Milliken-Taylor ultrafilter, and more.

A general formula for Fubini products was given by Todorcevic-Dobrinen and Milovich [19, 35], but more relevant to out needs, the following upper bound for Fubini sums:

Theorem 3.9. *For any two ultrafilters U,W_n on ω,*

$$\sum_U W_n\leq_T U\times\prod_{n<\omega}W_n.$$

Let us reproduce Kanamori's result on ω:

Theorem 3.10. *Suppose that U and W_n are (ω_1,ω_1)-cohesive for every $n<\omega$. Then $\sum_U W_n$ is (ω_1,ω)-cohesive.*

Proof. Since (λ,μ)-cohesiveness is downwards closed with respect to the Tukey order, by theorem 3.9, it remains to see that $U\times\prod_{n<\omega}W_n$ is (ω_1,ω)-cohesive. In the next lemma we sill prove that the Cartesian product of ω-many (ω_1,ω_1)-cohesive directed sets is (ω_1,ω)-cohesive. □

Lemma 3.11. *Suppose that $\{\mathbb{P}_i\mid i<\omega\}$ is a countable set of (ω_1,ω_1)-cohesive directed sets. Then $\prod_{n<\omega}\mathbb{P}_i$ is (ω_1,ω)-cohesive*

Proof. Let $\langle(p_{\alpha,m})_{m<\omega}\mid\alpha<\omega_1\rangle\subseteq\prod_{m<\omega}\mathbb{P}_m$. Inductively find $X_0\supseteq X_1\supseteq X_2$... all of size ω_1 such that for every m, $\{p_{\alpha,m}\mid\alpha\in X_m\}$ is bounded by p^0_m. Choose $\alpha_i\in X_i$ so that $i\neq j$ implies $\alpha_i\neq\alpha_j$. Let p^*_m be an extension of p^0_m and $p_{\alpha_i,m}$ for $i<m$, which exists by directedness. Then we claim that $\{(p_{\alpha_i,m})_{m<\omega}\mid i<\omega\}$ is bounded by $(p^*_m)_{m<\omega}$. Indeed, for every $i<\omega$, and every $m<\omega$, if $i<m$, then by p^*_m was chosen to be an extension of $p_{\alpha_i,m}$. If $i\geq m$, then $\alpha_i\in X_i\subseteq X_m$. Hence $p_{\alpha_i,m}$ is bounded by p^0_m and in turn by p^*_m. □

On $\kappa>\omega$, there is an analog formula: for U an ultrafilter on γ and $\langle W_\alpha\mid\alpha<\gamma\rangle$ any ultrafilters,

$$\sum_U W_\alpha\leq_T U\times\prod_{\alpha<\gamma}W_\alpha.$$

However the argument above does not generalize to arbitrary products, as we might not be able to find longer decreasing sequence in the course of the lemma.

4. Depth and point spectrum of ultrafilters

In this section we address the property of (λ,λ)-cohesivness for general λ.

Claim 4.1. *For any infinite cardinal λ, U is not $(cf(\lambda), cf(\lambda))$-cohesive iff $\lambda \leq_T U$*

Proof. Note that $\lambda \equiv_T cf(\lambda)$ and since $cf(\lambda)$ is regular, $[cf(\lambda)]^{<cf(\lambda)} \equiv_T (cf(\lambda),<)$, and the rest follows from Theorem 3.1. □

The following theorem is due to Kanamori:

Theorem 4.2 (Kanamori[30])**.** *Suppose that $2^\kappa = \kappa^+$, then any ultrafilter U over κ is not (κ^+,κ^+)-cohesive.*

The previous theorem is a special case of the following theorem since under $2^\kappa=\kappa^+$, $\mathfrak{ch}(U)=\kappa^+$ for any uniform ultrafilter on κ:

Theorem 4.3. *Let U be a uniform ultrafilter over κ. Then $\mathfrak{ch}(U) \leq_T U$.*

Proof. Let us construct a sequence of length $\mathfrak{ch}(U)$ witnessing that $\mathfrak{ch}(U) \leq_T U$. Let $\mathcal{B}$ be a base for U of size θ, and let us construct a sequence $\langle b^*_i \mid i < \mathfrak{ch}(U)\rangle$ starting with $b^*_0 = b_0$. Suppose that $\langle b^*_i \mid i<\gamma\rangle$ has been defined. By minimality of $\mathfrak{ch}(U)$, there is $X \in U$ such that for any $i<\gamma$, $b^*_i \not\subseteq X$. Since $\mathcal{B}$ is a base, there is $b \in \mathcal{B}$ such that $b \subseteq X$, and therefore for any $i<\gamma$, $b^*_i \not\subseteq b$. let $i_\gamma < \mathfrak{ch}(U)$ be minimal such that $b^*_{i_\gamma}$ is not generated by the previous b_i's, and $b^*_\gamma = b_{i_\gamma}$. It is not hard to check that $\langle b^*_i \mid i<\mathfrak{ch}(U)\rangle$ is again a base for U. Suppose towards a contradiction that there is $I \subseteq \theta$ unbounded such that $\bigcap_{i\in I} b^*_i \in U$. Then there $j<\theta$ such that $b^*_j \subseteq \bigcap_{i\in I} b^*_i$. Pick any $i \in I$ such that $i>j$, then $b^*_j \subseteq b^*_i$, contradicting the choice of b^*_i. □

Kanamori asked [30, Question 2] the following:

Is the existence of a κ-complete ultrafilter over $\kappa>\omega$ which is (κ^+,κ^+)-cohesive consistent?

In this section we will answer Kanamori's question by studying the *point spectrum* and establishing some connections of it to the order $(U,\supseteq^*)$. This order was studied by Milovich [34] and later by Dobrinen and the author [4].

4.1. The point spectrum of an ultrafilter.

Define the *point spectrum* of an ultrafilter U by

$$Sp_T(U)=\{\lambda\in Reg \mid \lambda \leq_T (U,\supseteq)\}.$$

By Claim 4.1, we also have

$$Sp_T(U)=\{\lambda\in Reg \mid U \text{ is not } (\lambda,\lambda)\text{-cohesive}\}.$$

Isbell [28] and independently Juhász [29] proved that cardinals such that $\kappa^{<\kappa}=\kappa$ always admits a uniform ultrafilter U which is not $(2^\kappa,\omega)$-cohesive, and therefore $Sp(U) = Reg \cap [\omega, 2^\kappa]$. Moreover, if for example, κ is κ-compact (or even less- see [14]) there is always a κ-complete ultrafilter, and even one which extends the club filter, which is not $(2^\kappa, \kappa)$-cohesive. Such ultrafilter in particular are not (λ, λ)-cohesive for any regular $\kappa \leq \lambda \leq 2^\kappa$.

Definition 4.4. An ultrafilter U is $(\lambda, \mu)^*$-cohesive if the directed set $(U, \supseteq^*)$ (λ, μ)-cohesive; that is, if for every sequence $\langle A_\alpha \mid \alpha < \lambda\rangle \subseteq U$ there is $I \in [\lambda]^\mu$ such that $\{A_i \mid i \in I\}$ admits a pseudo intersection in U

Similar to the usual cohesiveness characterization, we have the following:

Proposition 4.5. *Let U be an ultrafilter. The following are equivalent:*

1. *For any μ-directed poset $\mathbb{P}$, such that $|\mathbb{P}| \leq \lambda$, $\mathbb{P} \leq_T (U, \supseteq^*)$.*
2. *$([\lambda]^{<\mu}, \subseteq) \leq_T (U, \supseteq^*)$.*
3. *U is not $(\lambda, \mu)^*$-cohesive.*

Define

$$Sp^*_T(U) = \{\lambda \in Reg \mid \lambda \leq_T (U, \supseteq^*)\} = \{\lambda \in Reg \mid U \text{ is not } (\lambda, \lambda)^*\text{-cohesive}\}.$$

It is easy to see that $(U, \supseteq^*) \leq_T (U, \supseteq)$ and that (λ, μ)-cohesivness implies $(\lambda, \mu)^*$-cohesivness. Moreover the other implication is usually true as well:

Lemma 4.6. *Let U be a uniform ultrafilter over κ and a cardinal μ such that $cf(\mu) \neq \kappa$. If U is $(\lambda, \mu)^*$-cohesive then U is (λ, μ)-cohesive.*

Proof. Assume that U is $(\lambda, \mu)^*$-cohesive and let $\langle X_i \mid i < \lambda\rangle \subseteq U$. By assumption, there is $A \in U$ and $I \in [\lambda]^\mu$ such that for every $i \in I$, $A \subseteq^* X_i$. For each $i \in I$, let $\xi_i < \kappa$ be such that $A \setminus \xi_i \subseteq X_i$. Let us split into cases. If $\mu > \kappa$, then there is $I' \in [I]^\mu$ and $\xi^* < \kappa$ such that for every $i \in I'$, $\xi_i = \xi^*$. If $cf(\mu) < \kappa$, and let $\langle \mu_i \mid i < cf(\mu)\rangle$ be regular cardinals different from κ converging to μ. Write $I = \biguplus_{i<cf(\mu)} I_i$ where $|I_i| = \mu_i$, for each $i < cf(\mu)$. If $\mu_i < \kappa$, we let $\eta_i = \sup_{j\in I_i} \xi_j$, and if $\mu_i > \kappa$, since it is regular, we can apply the previous part to find $J_i \subseteq I_i$, $|J_i| = \mu_i$ such that for every $j \in J_i$, $\xi_j = \eta_i$. Then we can take $\xi^* = \sup_{i<cf(\mu)} \eta_i < \kappa$ and we let $I' = \bigcup_{i<cf(\mu)} J_i$. In any case, $A \setminus \xi^* \subseteq \bigcap_{i\in I'} X_i$. By uniformity, $A \setminus \xi^* \in U$ and therefore $\bigcap_{i\in I'} X_i \in U$. □

Corollary 4.7. *If $cf(\lambda) \neq \kappa$, then U is (λ, λ)-cohesive iff U is $(\lambda, \lambda)^*$-cohesive. In particular,*

$$Sp_T(U) \setminus \{\kappa\} = Sp^*_T(U) \setminus \{\kappa\}$$

Remark 4.8. For any uniform ultrafilter over κ will have $\kappa \in Sp_T(U)$. However U can be $(\kappa, \kappa)^*$-cohesive if for example U is a p-point. In fact, for κ-complete ultrafilters U over κ, U is a p-point iff U is $(\kappa, \kappa)^*$-cohesive. Or in other words, for κ-complete ultrafilters, U is not a p-point iff $Sp_T(U) = Sp^*_T(U)$.

Definition 4.9. Let U be an ultrafilter. Let the *Depth Spectrum of U* be denoted by $Sp_{dp}(U)$ and defined to be the set of all regular cardinal lengths θ of sequences $\langle A_i \mid i<\theta\rangle \subseteq U$ which are $\supseteq^*$-decreasing and have no measure one pseudo intersection in U. Define the *Depth* of U to be $\mathfrak{dp}(U)=\min(Sp_{dp}(U))$.

It is not hard to prove using Zorn's lemma that $Sp_{dp}(U)\neq\emptyset$ and therefore $\mathfrak{dp}(U)$ is well defined. Note that if θ is singular, and $\langle A_i \mid i<\theta\rangle$ is $\supseteq^*$-decreasing with no $\supseteq^*$-bound, then $cf(\theta)\in Sp_{dp}(U)$. Hence $\mathfrak{dp}(U)$ is a regular cardinal.

Suppose that κ is κ-compact[3], and let $\langle X_i \mid i<\theta\rangle$ be a tower. Since any tower has the $<\kappa$-intersection property, there is a κ-complete ultrafilter U such that for every i, $X_i\in U$. Hence $\theta\in Sp_{dp}(U)$. In particular, the tower number $\mathfrak{t}_\kappa\in Sp_{dp}(U)$ for some U. Since U is uniform, then it has to be that $\mathfrak{t}_\kappa=\mathfrak{dp}(U)$ as any pseudo intersection in U must have size κ.

Proposition 4.10. *$Sp_{dp}(U)\subseteq Sp_T(U)$.*

Proof. Fix a witnessing sequence $\langle A_\alpha \mid \alpha<\theta\rangle\subseteq U$ is $\subseteq^*$-decreasing. If U was (θ,θ)-cohesive, then there would have been $I\in[\theta]^\theta$ such that $A\in U$ is a pseudo intersection of $\{A_i \mid i\in I\}$. We claim that A is a pseudo intersection for the entire sequence. Indeed, let $\alpha<\theta$, then there is $\alpha'\in I$ such that $\alpha'\geq\alpha$. Hence $A\subseteq^* A_{\alpha'}\subseteq^* A_\alpha$. Contradiction. □

The most general setup to examine the point spectrum is the Galois-Tukey connections (see for example Blass's Chapter in [15]). However, this generality will not contribute to our specific interest in ultrafilters. Given a directed set $\mathbb{P}$, the *lower character* $l(\mathbb{P})$ is the smallest size of an unbounded family in $\mathbb{P}$, while the *upper character* $u(\mathbb{P})$ is the smallest size of a cofinal subset of $\mathbb{P}$. In our case, where $\mathbb{P}=(U,\supseteq)$ or $\mathbb{P}=(U,\supseteq^*)$ we get:

Proposition 4.11. *For any uniform ultrafilter U, we have:*

1. $u((U,\supseteq))=u((U,\supseteq^*))=\mathfrak{ch}(U)$.
2. $l(U,\supseteq)=crit(j_U)$ *is the completeness degree of the ultrafilter U.*
3. $l(U,\supseteq^*)=\mathfrak{dp}(U)$

We have that $crit(j_U)\leq\mathfrak{dp}(U)\leq\mathfrak{ch}(U)$.

Proof. $(1),(2)$ are well known facts. To see (3), by minimality, $l(U,\supseteq^*)\leq\mathfrak{dp}(U)$. The other direction follows from the next simple lemma which implies that for every sequence of length $\theta<\mathfrak{dp}(U)$ is $\supseteq^*$-bounded. □

Lemma 4.12. *For any sequence $\langle X_i \mid i<\theta\rangle\subseteq U$ for $\theta\leq\mathfrak{dp}(U)$, there is $\langle X^*_i \mid i<\theta\rangle\subseteq U$ such that $X^*_i\subseteq X_i$ for all i which is $\subseteq^*$-decreasing.*

Proof. We construct X_i by induction. At successor step, we let $X^*_{i+1}=X_{i+1}\cap X^*_i$. At limit steps α, since $\alpha<\mathfrak{dp}(U)$, the sequence $\langle X^*_i \mid i<\alpha\rangle$ which by induction is $\subseteq^*$-decreasing has a $\subseteq^*$-lower bound $A\in U$. We let $X^*_\alpha=A\cap X_\alpha$. □

[3] That is, every κ-cpomplete filter can be extended to a κ-complete ultrafilter.

Recall that an ultrafilter U over $\kappa \geq \omega$ is called a P_λ-point, if $(U, \supseteq^*)$ is λ-directs. A p-point is a P_{κ^+}-point.

Corollary 4.13. *$\mathfrak{dp}(U)$ is the unique λ such that U is P_λ-point but not a P_{λ^+}-point.*

Most of the following propositions regarding $Sp_T(U)$, can be derived from the general set up:

Proposition 4.14. *Let U be a uniform ultrafilter, then:*

1. $\mathfrak{dp}(U) \leq cf(\mathfrak{ch}(U)) \in Sp_T(U)$.
2. $\min(Sp_T(U)) = crit(j_U), \min(Sp_T(U, \supseteq^*)) = \mathfrak{dp}(U)$.
3. $\mathfrak{ch}(U)$ *is an upper bound for* $Sp_T(U)$.

Proof. For (1), we have already seen that $cf(\mathfrak{ch}(U)) \in Sp_T(U)$ by Theorem 4.3. Suppose otherwise that $cf(\mathfrak{ch}(U)) < \mathfrak{dp}(U)$ and fix $\langle \theta_j \mid j < cf(\mathfrak{ch}(U)) \rangle$ cofinal in $\mathfrak{ch}(U)$. Let $\langle b_i \mid i < \mathfrak{ch}(U) \rangle$ be a base for U. For each θ_j, by minimality, there is $x_j \in U$ which is not $\subset^*$-generated by $\langle b_i \mid i < \theta_j \rangle$. Since $cf(\mathfrak{ch}(U)) < \mathfrak{dp}(U) = l(U, \supseteq^*)$, the sequence $\langle x_j \mid j < cf(\mathfrak{ch}(U)) \rangle$ has a pseudo-intersection $x^* \in U$. Then x^* cannot be $\subseteq^*$-generated by any base element, contradiction.

For (2), if $\theta < crit(j_U)$, or $\theta < \mathfrak{dp}(U) = l(U, \supseteq^*)$, then $\theta \notin Sp_T(U)$ or $\theta \notin Sp_T(U, \supseteq^*)$ respectively, as any sequence of length θ is bounded. To see for example that $\mathfrak{dp}(U) \in Sp_T(U, \supseteq^*)$ (the proof that $crit(j_U) \in Sp_T(U)$ is completely analogous), we note that $\mathfrak{dp}(U) \in Sp_{dp}(U) \subseteq Sp_T(U)$. To see (3), let $\mathfrak{ch}(U) < \lambda$ be regular. Then given any λ-many sets in U, λ-many of them must contain the same element from a fixed base of size $\mathfrak{ch}(U)$. Hence U will be (λ, λ)-cohesive, namely $\lambda \notin Sp_T(U)$. □

Corollary 4.15. *If $\mathfrak{dp}(U) \neq \kappa$, then $\min(Sp_T(U) \setminus \{\kappa\}) = \mathfrak{dp}(U)$.*

Corollary 4.16. *If $\mathfrak{ch}(U)$ is regular, then $\mathfrak{ch}(U) = \max(Sp_T(U))$.*

Note that using Theorem 3.1, we can characterize in terms of the ultrapower $\sup Sp_T(U)$ as the least μ such that for every $\lambda > \mu$ regular, M_U does not have a thin cover for $j_U''\lambda$. Hence the previous corollary provides an ultrapower characterization of $\mathfrak{ch}(U)$, whenever this cardinal is regular. Ultrafilters with a singular character exists, for example, if we add $\kappa^{+\kappa^+}$-many Cohen function to κ in a model of GCH, then in the extension the $2^\kappa = \kappa^{+\kappa^+} = \mathfrak{ch}(U)$ for all U on κ. However, in the model above we will still have that $\mathfrak{ch}(U) = \sup(Sp_T(U))$ and therefore the proposed ultrapower characterization remains valid. Hence the following question is natural:

Question 4.17. Is it true that $\mathfrak{ch}(U) = \sup(Sp_T(U))$ for any uniform ultrafilter U? how about κ-complete ultrafilters over κ?

In that direction Isbell proved [28] then if every singular cardinal is strong limit then for every $\mathbb{P}$, $u(\mathbb{P}) = \sup(Sp_T(U))$. However, this theorem is irrelevant for us, since if $\mathfrak{ch}(U)$ is singular then $\kappa < \mathfrak{ch}(U) \leq 2^\kappa$, and Isbell's theorem does not apply.

One my wonder what is the possible cofinalities of $\mathfrak{ch}(U)$. It is not hard to prove that for κ-complete ultrafilters $cf(\mathfrak{ch}(U)) \geq \kappa^+$. If we give up κ-completeness we will see in Proposition 5.6 that small cofinality is consistent.

Proposition 4.18. (1) *For any two κ-complete ultrafilters over $\kappa > \omega$, $Sp_T(U \cdot W) = Sp_T(U) \cup Sp_T(W)$.*
(2) *For any two ultrafilter U, W on ω such that $W \cdot W \equiv_T W$, $Sp_T(U \cdot W) = Sp_T(U) \cup Sp_T(W)$.*

Proof. We prove (1), (2) together, since both $U, W \leq_T U \cdot W$, $Sp_T(U \cdot W) \supseteq Sp_T(U) \cup Sp_T(W)$. In the other direction, if $\lambda \in Sp_T(U \cdot W)$, then $U \cdot W$ is not (λ, λ)-cohesive. By Theorem 3.7 for (1), or Corollary 3.8 for (2), either U or W are not (λ, λ)-cohesive, namely, $\lambda \in Sp_T(U) \cup Sp_T(W)$. □

This cannot be improved, even just for p-point. Indeed, it is consistent that there are p-point W, U on ω such that $\mathfrak{ch}(W), \mathfrak{ch}(U) < \mathfrak{d}$ (see the remark following [19, Cor. 36] and other constructions where $U \not\geq_T \omega^\omega$), and $\mathfrak{d}$ is regular. By Milovich [35], $U \cdot W \equiv_T U \times W \times \omega^\omega$ and therefore $\mathfrak{d} \leq_T U \cdot W$. On the other hand, $\mathfrak{d} \notin Sp_T(U) \cup Sp_T(W)$ as both $Sp_T(U)$ and $Sp_T(W)$ are bounded by $\max(\mathfrak{ch}(U), \mathfrak{ch}(W)) < \mathfrak{d}$.

4.2. Strong generating sequences and P_λ-points.

Definition 4.19. Let U be an ultrafilter over κ. A sequence $\langle A_\alpha \mid \alpha < \lambda \rangle$ such that:

(1) If $\alpha < \beta < \lambda$ then $A_\beta \subseteq^* A_\alpha$.
(2) For every $X \in U$ there is $\alpha < \lambda$ such that $A_\alpha \subseteq^* X$.

is called a *strong generating sequence* for U.

Remark 4.20. For example, if U is a p-point and $2^\kappa = \kappa^+$ then U has a strong generating sequence of length κ^+.

Garti and Shelah (see [17] for details) noticed that assume κ is supercompact and $\lambda > \kappa$ is regular, it is consistent that there is a κ-complete ultrafilter U over κ with a strong generating sequence of length λ.

Lemma 4.21. *If U has a generating sequence of length λ, then $\lambda > \kappa$, and U is a P_λ point. Moreover, $\mathfrak{dp}(U) = \lambda$.*

Proof. Let $\langle A_\alpha \mid \alpha < \lambda \rangle$ be a strong generating sequence. Given any sequence $\langle X_\alpha \mid \alpha < \theta \rangle$ where $\theta < \lambda$, for every $\alpha < \theta$ there is $\beta_\alpha < \lambda$ such that $A_{\beta_\alpha} \subseteq^* X_\alpha$. Take $\beta^* = \sup_{\alpha < \theta} \beta_\alpha$, then A_{β^*} $\supseteq^*$-bounds the X_α's. For the "Moreover" part, by Corollary 4.13 $\lambda \leq \mathfrak{dp}(U)$. As a generating sequence, $\langle A_\alpha \mid \alpha < \lambda \rangle$ cannot have a pseudo intersection. Hence $\lambda \in Sp_{dp}(U)$, and the equality holds. □

Corollary 4.22. *If U has a strong generating sequence of length λ, then $\mathfrak{dp}(U) = \mathfrak{ch}(U)$ and*

$$Sp_{dp}(U) = Sp_T(U) \setminus \{\kappa\} = \{\lambda\}.$$

Proof. By Corollary 4.15 and Proposition 4.14,

$$\kappa < \lambda = \mathfrak{dp}(U) = \min(Sp_T(U) \setminus \{\kappa\}) \leq \sup(Sp_T(U)) \leq \mathfrak{ch}(U) \leq \lambda.$$

Hence

$$\{\lambda\} \subseteq Sp_{dp}(U) \subseteq Sp_T(U) \setminus \{\kappa\} = \{\lambda\}.$$

□

The point spectrum characterizes the existence of strong generating sequences:

Lemma 4.23. *Let U be a uniform ultrafilter on κ. U has a strongly generating sequence of length λ if and only if $\mathfrak{dp}(U) = \mathfrak{ch}(U) = \lambda$.*

Proof. One direction follows from Corollary 4.22. In the other direction, suppose that $\mathfrak{dp}(U) = \mathfrak{ch}(U) = \lambda$, given a base $\langle X_i \mid i < \lambda \rangle$ for U, we may apply Lemma 4.12 to the base to obtain a strong generating sequence for U of length λ. □

Corollary 4.24. *If κ is supercompact cardinal, it is consistent that κ carries a κ-complete ultrafilter which is (κ^+, κ^+)-cohesive.*

In the next section we address the question of consistency strength of such ultrafilters.

5. Consistency results

Although the existence of a long generating sequence seems stronger than Kanamori's question, it is actually equivalent:

Theorem 5.1. *For any $\lambda > \kappa$ The following are equicosistent:*

1. *there exists of an ultrafilter with a strong generating sequence of length λ.*
2. *there is a κ-complete ultrafilter U with $\min(Sp_T(U) \setminus \{\kappa\}) = \lambda$; U is (μ, μ)-cohesive for every regular $\mu \in (\kappa, \lambda)$.*
3. *there exists of a κ-complete ultrafilter U with $\mathfrak{dp}(U) = \lambda$.*
4. *There exists a P_{λ^+}-point over κ, and λ is regular.*

Proof. (1) implies (2) by Corollary 4.22. To see that (2) implies (3), let U witness (2), and let $U_0 \leq_{RK} U$ be normal. Then $Sp^*_T(U_0) \subseteq Sp^*_T(U)$, and $\mathfrak{dp}(U_0) > \kappa$. Hence $\mathfrak{dp}(U_0) \geq \min(Sp_T(U) \setminus \{\kappa\}) = \lambda$. To see that (3) implies (4), by 4.13, U is a P_{λ^+}-point and we already made the observation that $\mathfrak{dp}(U)$ must be regular. Finally for (4) implies (1), suppose that U is a P_{λ^+}-point for some regular $\lambda > \kappa$, then U a κ-complete ultrafilter. Levi collapsing $\mathfrak{ch}(U)$ to be λ (if needed) is a λ-closed forcing, and does not introduce new $< \lambda$-sequence. Hence U is still a κ-complete ultrafilter in the extension. Moreover, we have $\mathfrak{dp}^{V[G]}(U) = \lambda = \mathfrak{ch}^{V[G]}(U)$. Hence by Lemma 4.23, U had a strong generating sequence in $V[G]$. □

In particular, the existence of a κ-complete (κ^+,κ^+)-cohesive ultrafilter is equiconsistent with the existence of an ultrafilter having a strong generating sequence of length κ^{++}- a principle which we conjecture having higher consistency strength than $o(\kappa)=\kappa^{++}$.

Next we would like to treat the possible complexity of $Sp_T(U)$. First, we would like to rise the following question:

Question 5.2. Is it consistent that $Sp_T(U)$ is not an interval of regular cardinals?

A natural approach would be to take two ultrafilter U, W such that U has a strong generating sequence of length κ^+ (for example) and W has a strong generating sequence of length κ^{+++}. By proposition 4.18 $Sp_T(U\cdot W)=Sp_T(U)\cup Sp_T(W)=\{\kappa^+,\kappa^{+++}\}$. We do not know if it is consistent to have two such ultrafilters. However, the usual way to obtain such ultrafilters; that is, to iterate Mathias forcing λ-many times is doomed. Indeed if we iterate the above forcing to produce one ultrafilter with a strong generating sequence of length λ, and then iterate again λ'-many times to generate an ultrafilter with a strong generating sequence of length λ'. If λ,λ' have different cofinalities, any ultrafilter in the generic extension strongly generated by $cf(\lambda)$-many sets would have to be generated from an intermediate step of the iteration. The following lemma assures that adjoining even one Mathias generic set prevents ultrafilters from being generated by the ground model.

Lemma 5.3. *Let a_G be an $\mathbb{M}_U$-generic set. Denote by*

$$R=\bigcup_{i<\kappa}[(a_G)_{2i},(a_G)_{2i+1}).$$

Then both R and $\kappa\setminus R$ cannot contain an unbounded set from the ground model.

Proof. Suppose otherwise that $X\in V$ and $X\subseteq R$ (the proof for $\kappa\setminus R$ is similar). Let $\langle a,A\rangle\in G$ be a condition such that $\langle a,A\rangle\Vdash_{\mathbb{M}_U}\check{X}\subseteq\underset{\sim}{R}$. We may assume that $\max(a)=a_{2i+1}$ for some $i<\kappa$ (namely $otp(a)$ is an even successor ordinal). Since X is unbounded, there is $x\in X\setminus a_{2i+1}+1$, and since $A\in U$, there is $\alpha\in A\setminus X$. extend $\langle a,A\rangle$ to $\langle a\cup\{\alpha\},A\setminus\alpha+1\rangle$. This condition forces that $x\in X\setminus R$, contradiction. □

Lemma 5.4. *Let $V[G]$ be the usual Cohen extension where we added κ^{++}-many Cohen generic functions from κ to 2 to a model of $2^\kappa=\kappa^+$. Then any uniform ultrafilter $U\in V[G]$ is not (κ^{++},κ^+)-cohesive. In particular it is not (κ^+,κ^+)-cohesive and not $(\kappa^{++},\kappa^{++})$. Namely $Sp_T(U)\supseteq\{\kappa,\kappa^+,\kappa^{++}\}$.*

Proof. Let G be V-generic for $Add(\kappa,\kappa^{++})$. Let $U\in V[G]$ be an ultrafilter and let $\langle X_\alpha\mid\alpha<\kappa^{++}\rangle$ be the Cohen mutually generic subsets of κ added by G. Note that for every α, either $X_\alpha\in U$ or $X^c_\alpha\in U$. Since the complement of a Cohen generic set is also Cohen generic, we may assume that $\langle X_\alpha\mid\alpha<$

$\kappa^{++}\rangle \subseteq U$. Let us prove that the sequence of Cohen generics witnesses that U is not (κ^{++},κ^+)-cohesive. Suppose towards a contradiction that for some I of size κ^+, $Y=\bigcap\{X_i \mid i\in I\}\in U$. In particular Y is unbounded in κ (since U is uniform). Since $Y\subseteq\kappa$, by κ^+-c.c, there is $J\subseteq\kappa^{++}$ such that $|J|=\kappa$ and $Y\in V[G\restriction J]$. Pick any $\beta\in I\setminus J$ which exists since $|I|>|J|$. By mutual genericity, X_i is generic over $V[G\restriction J]$, in particular X_i cannot contain a set of size κ from $V[G\restriction J]$, contradicting $Y\subseteq X_i$. □

By Woodin, starting with a model of GCH and a measurable κ with Mitchell order $o(\kappa)=\kappa^{++}$, it is possible to get a generic extension in which κ measurable and κ^{++}-many mutually generic Cohen functions (over some intermediate model where $2^\kappa=\kappa^+$).

Corollary 5.5. *In the model above, for every ultrafilter U, $\mathfrak{dp}(U)<\mathfrak{ch}(U)$.*

Clearly, the previous argument generalized for $\lambda>\kappa^{++}$ to produce a model where $2^\kappa=\lambda$ and for every uniform ultrafilter U over κ, U is not (λ,κ^+)-cohesive (so also all the instance of (ρ,μ)-cohesiveness fail for $\kappa^+\leq\mu\leq\rho\leq\lambda$). This improves Theorem 4.2 from [8], which proves this consistency (with the same model!) only for normal ultrafilters. The argument from the previous lemma can be used to give an example of an ultrafilter with a character of small cofinality:

Proposition 5.6. *Relative to a measurable cardinal, it is consistent that there is a uniform ultrafilter on a regular cardinal κ with $cf(\mathfrak{ch}(U))=\omega_1$*

Proof. Raghavan and Shelah [37] proved that after adding $\kappa^{+\omega_1}$-many Cohen reals, we can find a uniform ultrafilter D in the extension with $\mathfrak{ch}(D)\leq\kappa^{+\omega_1}$. Let us prove that in fact $\mathfrak{ch}(D)=\kappa^{+\omega_1}$. We think of the forcing $Add(\kappa^{+\omega_1},\omega)$ as adding $\kappa^{+\omega_1}$-many characteristic sets to κ $\langle X_\alpha \mid \alpha<\kappa^{+\omega_1}\rangle$ with finite approximation. The ultrafilter D have to pick for each $\alpha<\kappa^{+\omega_1}$ either $X_\alpha\in D$ or $\kappa\setminus X_\alpha\in D$. As in the proof of 5.4, we may assume that for every $\alpha<\kappa^{+\omega_1}$, $X_\alpha\in D$. If $\mathfrak{ch}(D)<\kappa^{+\omega_1}$, we could have found ω_1-many sets X_α for $\alpha\in I$ such that $\bigcap_{\alpha\in I}X_\alpha\in D$. Let Y be any countable subset of $\bigcap_{\alpha\in I}X_\alpha$. Using the chain condition, we see that Y belongs the extension by countably many of the Cohen reals, and therefore to the extension by countably many of the X_α's. That is $Y\in V[\langle X_\alpha \mid \alpha\in J\rangle]$ where $J\in[I]^\omega$. Pick any $\alpha^*\in I\setminus J$. Then X_α is generic over $V[\langle X_\alpha\mid\alpha\in J\rangle]$. However, by genericity, the set X_{α^*} is only finitely approximated in $V[\langle X_\alpha\mid\alpha\in J\rangle]$ which produces a contradiction. □

Question 5.7. Is there a strong limit regular cardinal κ carrying a uniform ultrafilter U such that $cf(\mathfrak{ch}(U))<\kappa$?

What about $Sp_{dp}(U)$ in the Cohen extension? Alan Dow informed us that Kunen proved in his master thesis that not only there are no towers of length $\aleph_2$, but there are no $\subseteq^*$ chains of ordertype $\aleph_2$ in the extension by the finite support product of $\aleph_2$-many reals. He also suggested a simpler argument for towers which generalizes to regular uncountable cardinals. We need a slight strengthening of that argument to show that Cohen forcing does not add chains modulo bounded of length κ^{++} to a normal ultrafilter U which are unbounded in $(U, \supseteq^*)$.

Lemma 5.8. *Suppose $2^\kappa = \kappa^+$. After forcing with $Add(\kappa, \kappa^{++})$, there is no normal ultrafilter with a sequence $\langle A_i \mid i < \kappa^{++}\rangle \subseteq U$ which is $\subseteq^*$-decreasing, and unbounded in $(U, \subseteq^*)$. Equivalently, $\kappa^{++} \notin Sp_{dp}(U)$.*

Proof. Suppose otherwise and let $\dot{\mathcal{U}}, \dot{\mathcal{A}}$ be names and $p \in Add(\kappa, \kappa^{++})$ forcing all the relevant information. We may assume that $\mathcal{A} = \langle \dot{A}_i \mid i < \kappa^{++}\rangle$ and $\mathcal{U} = \langle \dot{X}_i \mid i < \kappa^{++}\rangle$ are sequences of nice names, and p forces that $\mathcal{U}$ is an ultrafilter and $\mathcal{A}$ is $\subseteq^*$-decreasing and unbounded in $(\mathcal{U}, \subseteq^*)$. Let $M \prec H_\theta$ be an elementary submodel of size κ^+, closed under κ-sequence with

$$p, \langle \dot{X}_i \mid i < \kappa^{++}\rangle, \langle \dot{A}_i \mid i < \kappa^{++}\rangle ... \in M.$$

Also assume that $M \cap \kappa^{++} = \delta \in \kappa^{++}$. First we note that for each $i < \delta$, $\dot{X}_i, \dot{A}_i \in M$ is a nice name, and since M is closed under κ-sequnces and by κ^+-cc, $\dot{X}_i$ and $\dot{A}_i$ are in fact names of $Add(\kappa, \delta)$.

Claim 5.9. *In $V[G \restriction \delta]$ we have:*

(1) *$\{(\dot{X}_i)_{G\restriction\delta} \mid i < \delta\}$ is the ultrafilter $\mathcal{U}_\delta = (\dot{\mathcal{U}})_G \cap V[G \restriction \delta]$*
(2) *$\mathcal{U}_\delta$ is normal.*
(3) *$\langle (\dot{A}_i)_{G\restriction\delta} \mid i < \kappa^{++}\rangle$ is a tower in $\mathcal{U}_\delta$.*

□

Proof of Claim. Clearly, $\{(\dot{X}_i)_{G\restriction\delta} \mid i < \delta\} \subseteq \mathcal{U}_\delta$. Hence it suffices to prove that $\{(\dot{X}_i)_{G\restriction\delta} \mid i < \delta\}$ is an ultrafilter in $V[G \restriction \delta]$. Suppose not, and let $\dot{X}$ be a nice name such that for some $q \geq p$, $q \in G \restriction \delta$,

$$q \Vdash_{Add(\kappa,\delta)} \dot{X}, \kappa \setminus \dot{X} \notin \{\dot{X}_i \mid i < \delta\}.$$

Again, by closure of M to κ-sequences and the κ^+-cc, $\dot{X}, q \in M$. Hence for every $i < \delta$, $M \models q \Vdash \dot{X} \neq \dot{X}_i \wedge \kappa \setminus \dot{X} \neq \dot{X}_i$. Since $\kappa^{++} \cap M = \delta$,

$$M \models q \Vdash \text{“}\dot{\mathcal{U}} \text{ is not an ultrafilter”},$$

contradiction. Normality just follows from the κ^+-cc and the fact that cofinality of δ is κ^+. A similar argument shows that $\langle (\dot{A}_i)_G \mid i < \kappa^{++}\rangle$ is a tower in $\mathcal{U}_\delta$. □

Note that $\dot{A}_\delta$ is forced by p to be $\subseteq^*$-bound, so by κ^+-cc, over the model $V[G \restriction \delta]$ we can find κ-many coordinates $I \subseteq [\delta, \kappa^{++})$ such that in $V[G \restriction \delta][G \restriction I]$ we will have a $\subseteq^*$-bound. Note that $G \restriction I$ is forcing equivalent to the extension by a single Cohen $Add(\kappa, 1)$. Hence it suffices to prove the following claim:

Claim 5.10. *Suppose that $\mathcal{W}$ is a normal ultrafilter (p-point is enough) on κ, $\langle A_i \mid i < \kappa^+\rangle$ is a tower in $\mathcal{W}$. Then adding a single Cohen function to κ does not add pseudo intersection X for $\langle A_i \mid i < \kappa^+\rangle$ which is positive with respect to the filter generated by $\mathcal{W}$ in the extension.*

Proof of claim. Suppose otherwise, the $\dot{A}$ is a name and p a condition such that

$$p \Vdash \text{“}\dot{A} \in (\overline{\mathcal{W}})^+ \text{ is a pseudo intersection of } \langle A_i \mid i < \kappa^+\rangle\text{”},$$

where $\overline{\mathcal{W}}$ is the filter generated by $\mathcal{W}$ in the extension. For each $q \geq p$ let $A_q = \{\alpha < \kappa \mid \exists r \geq q,\ r \Vdash \alpha \in \dot{A}\}$. Then A_q is in the ground model cover of $\dot{A}$, hence positive for $\mathcal{W}$. Therefore $A_q \in \mathcal{W}$. Since $Add(\kappa, 1)$ has size κ, and $\mathcal{W}$ is assumed to be a p-point, there is $A^* \in \mathcal{W}$ which is a pseudo intersection for all the A_q's. Since A_i is a tower, there is $i^* < \kappa^+$ such that $A^* \setminus A_i$ is unbounded in κ. Find $p_0 \geq p$ such that $p_0 \Vdash \dot{A} \setminus \xi \subseteq A_i$. Find ξ' such that $A^* \setminus \xi' \subseteq A_{p_0}$ and find $\rho > \xi, \xi'$ such that $\rho \in A^* \setminus A_i$. Hence $\rho \in A_{p_0}$ and by definition there is $p' \geq p_0$ forcing that $\rho \in \dot{A}$. But $\rho \notin A_i$ and p' is suppose to force also that $\dot{A} \setminus \xi \subseteq A_i$, contradiction. □

The following is due to Gitik:

Theorem 5.11. *The consistency strength of the existence of a $P_{\kappa^{++}}$-point is at least $o(\kappa) = \kappa^{++} + \kappa^+$.*

Question 5.12. What is the consistency strength of having $\kappa^{++} \in Sp_{dp}(U)$ for a κ-complete ultrafilter U?

Question 5.13. What is the consistency strength of the existence of a P_λ-point for $\lambda > \kappa^+$?

Equivalently,

Question 5.14. What is the consistency strength of the existence of a κ-complete (κ^+, κ^+)-cohesive ultrafilter over κ?

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