NON-NORMAL MAGIDOR-RADIN TYPES OF FORCINGS

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Abstract. We develop the non-normal variations of two classical Prikry-type forcings; namely, Magidor and Radin forcings. We generalize the fact that the non-normal Prikry forcing is a projection of the extender-based to a coordinate of the extender to our forcing and the Radin/Magidor-Radin-extender-based forcing from [Mer11, Mer03a]. Then, we show that both the non-normal variation of Magidor and Radin forcings can add a Cohen generic function to every limit point of cofinality ω of the generic club. Second, we show that these phenomenon is limited to the cases where the forcings are not designed to change the cofinality of a measurable κ to ω_1. Specifically, in the above-mentioned circumstances these forcings do not project onto any κ-distributive forcing. We use that to conclude that the extender-based Radin/Magidor-Radin forcing does not add fresh subsets to κ as well. In the second part of the paper we focus on the natural non-normal variation of Gitik's forcing from [Git86, §3]. Our main result shows that this poset can be employed to change the cofinality of a measurable cardinal κ to ω_1 while introducing a Cohen subset of κ.

1. Introduction

Singular Cardinal Combinatorics is a prominent area of research in modern set theory. The field is primarily concerned with the properties of singular cardinals and its small successors (such as $\aleph_\omega$ and $\aleph_{\omega+1}$) and how these change across the set-theoretic multiverse. During the last fifty years, research in this field have yielded some of the most sophisticated technologies ever invented in set theory. A paradigmatic example are the so-called *Prikry-type forcings*. The field was pioneered by Prikry [Pri70] who provided the first example of a forcing poset changing the cofinality of a measurable cardinal to $\aleph_0$ without collapsing cardinals. However, it was Magidor who through a series of groundbreaking discoveries [Mag76, Mag77a, Mag77b] placed Prikry-type posets in the spotlight. Other major results employing these forcings were obtained by Cummings and Woodin [Cum92a], Gitik [Git91, Git86] and Foreman and Woodin [FW91].

By nowadays Prikry-type forcings count with a beautiful and extensive theory, mostly developed and accounted by Gitik in [Git10]. During the last

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few decades the abstract study of Prikry-type forcings has became a topic of central interest in set theory. Like any other important mathematical structure $\mathcal{M}$, the problem of classifying the substructures of $\mathcal{M}$ is of high importance. In that direction, a problems which has elicited a major interest concerns the possible intermediate models of a generic extension by a Prikry-type forcing. This amounts to asking whether a given Prikry-type forcing $\mathbb{P}$ projects onto other Prikry-type posets or even onto other classical forcings – the epitome of these latter being Cohen forcing.

The following is a succinct account of what is known for Prikry forcing and its tree-like variations where these posets are defined using normal ultrafilters. First, Gitik, Koepke, and Kanovei [GKK10] proved that any intermediate model of a generic extension by Prikry forcing with a normal ultrafilter must be a generic extension by Prikry forcing (with the same normal ultrafilter). In contrast, Koepke, Rasch and Schlicht [KRS13] construted a Tree Prikry forcing yielding a minimal forcing extension – this phenomenon is akin to the classical *Sacks property* of Sacks forcing. More recently, Benhamou and Gitik [BG21], and afterwards Benhamou, Gitik, and Hayut [BGH23], proved that the classical Tree Prikry forcing with non-normal ultrafilters can project onto a wide variety of κ-distributive forcings of cardinality κ – including $\mathrm{Add}(\kappa, 1)$. In addition, that paper provides a non-trivial large-cardinal lower bound for this forcing to project onto every κ-distributive (even $<\kappa$-strategically closed) poset of cardinality κ. The results in [BGH23] are a sequel of a classical theorem of Gitik saying that Supercompact Prikry forcing can be arranged to project onto every κ-distributive forcing [Git10, §6.4] of cardinality κ. Finally, Benhamou and Gitik [BG23] constructed an ultrafilter U such that Prikry forcing with U projects onto $\mathrm{Add}(\kappa, \kappa^+)$, showing that the class of distributive forcings onto which the Tree Prikry forcing projects exceeds those of cofinality κ.

In the context of Magidor/Radin-like forcings – again, relative to normal ultrafilters – our knowledge is way more narrow. Fuchs [Fuc14] proved that if c, d are generic sequences for Magidor forcing of [Mag78] and $c \in V[d]$ then c is almost contained in d. Benhamou and Gitik [BG21, BG22a, BG22b] generalized Gitik-Koepke-Kanovei's result and provided a full characterization of the intermediate models of a generic extension by the Mitchell version from [Mit82] of Magidor forcing relative to a coherent sequence of measures with $o(\kappa) < \kappa^+$. Namely, if G is generic for the Magidor/Radin forcing then every intermediate model $V \subseteq M \subseteq V[G]$ is of the form $V[C]$ where C is a subset of the generic club added by G. In the case where $o(\kappa) < \kappa$, models of the form $V[C]$ are generic for a finite iteration of Magidor-like forcings.

The above results indicate that in the *normal context* one should not expect a rich variety of intermediate extensions for a given Prikry-type forcing, while in the *non-normal context* special constructions can provide a richer variety. This reflection invites to developing variations of the aforementioned forcings when the ultrafilters involved are non-normal.

In this paper we develop two new Prikry-type technologies – the Mitchell-style non-normal Magidor forcing and the non-normal Radin forcing, respectively. Versions of these forcings appeared somewhat implicitly in Merimovich's works on Extender-Based Magidor/Radin forcing [Mer03a, Mer11].

Let $\mathbb{M}[\vec{U}]$ (resp. $\mathbb{R}_u$) denote our non-normal version of the Mitchel-style Magidor (resp. Radin forcing) with respect to a generalized coherent sequence of ultrafilters[1] $\vec{U}$ (resp. a measure sequence u) of length ω_1. These two posets will be respectively developed in §3 and §4 of this paper. Later in §5 we shall employ them to demonstrate that they yield garden-variety of intermediate generic extensions. The mathematical meaning of this assertion is make precise by our first main theorem:

Main Theorem 1. *It it consistent for both $\mathbb{M}[\vec{U}]$ and $\mathbb{R}_u$ to yield a club $C \subseteq \kappa$ of cardinals with $\mathrm{otp}(C) = \omega_1$ such that every limit point of $\alpha \in C$ carries a Cohen generic function for $\mathrm{Add}(\alpha, 1)$.*

Therefore, in the above model, both $\mathbb{M}[\vec{U}]$ and $\mathbb{R}_u$ project onto $\mathrm{Add}(\alpha, 1)$ for every limit point $\alpha \in C$. This fact is optimal in the sense that one cannot hope for these forcings to project onto Cohen forcing $\mathrm{Add}(\alpha, 1)$ for a singular cardinal α of uncountable cofinality in the eventual Prikry-type extension. This conclusion will be inferred as a consequence of these posets not adding fresh subsets to κ (Corollary 5.10). Moreover, we show that the same conclusion is applicable Merimovich's Extender-Based Radin and Magidor/Radin forcings from [Mer03a, Mer11] (see Corollary 5.12).

So, is it possible for a Prikry-type forcing $\mathbb{P}$ to project onto $\mathrm{Add}(\kappa, 1)$ when κ is a measurable cardinal that changes its cofinality to ω_1 after forcing with $\mathbb{P}$? We show that the answer is affirmative but these requires fairly different methods to be established. Specifically, we show that the non-normal variation of Gitik's forcing $\mathbb{P}(\kappa, \omega_1)$ from [Git86] does the job. Unlike the previously mentioned posets, $\mathbb{P}(\kappa, \omega_1)$ changes the cofinality of a measurable cardinal κ with $o(\kappa) = \omega_1$ without introducing bounded subsets to κ. This poset is defined over a generic extension of V by an Easton-supported (a.k.a., *Gitik iteration*) of Prikry-type forcings (see [Git86]).

Our main result in regards to $\mathbb{P}(\kappa, \omega_1)$ reads as follows:

Main Theorem 2. *It is consistent for $\mathbb{P}(\kappa, \omega_1)$ to project onto $\mathrm{Add}(\kappa, 1)$.*

In a recent paper [GK24], Gitik and Kaplan have proved that certain iteration of Prikry-type forcings of length κ do not add fresh subsets to κ. In particular these results apply to the preparatory iteration in Theorem 2, which therefore do not add fresh sets to κ.

The structure of the paper is as follows. We begin with §2 discussing two non-normal variations of the classical notion of coherent sequence of normal ultrafilters. This analysis is used later in §3 where we present the Mitchel-styled Magidor forcing $\mathbb{M}[\vec{U}]$ with respect to a generalized coherent

[1]See Definition 2.1.

sequence $\vec{U}$. In this section we also show that this forcing can be recasted as a projection of Merimovich Extender-Based posets [Mer03a, Mer11]. In §4 we present the non-normal Radin forcing and in §5 we prove **Main Theorem** 1. In §6 we discuss the non-normal version of Gitik's forcing following [Git86] and prove **Main Theorem** 2. The manuscript is concluded with §7 by drawing possible future directions and proposing a few open questions.

Convention 1.1. Given U a κ-complete ultrafilter over κ we will tend to denote either by M_U or $\mathrm{Ult}(V,U)$ the transitive collapse of the ultrapower of V by U. Similarly, the induced elementary embedding from V to M_U will be denoted by j_U. When it comes to a forcing posets we shall stick to the *Israeli convention*; namely, when we write $p \le q$ we will be meaning that q is *stronger* (i.e., more informative) than p. Given a regular cardinal κ we shall denote by $\mathrm{Add}(\kappa,1)$ the *Cohen forcing* at κ; namely, conditions in $\mathrm{Add}(\kappa,1)$ are partial functions $p\colon \kappa \to 2$ with $|p| < \kappa$ ordered by $\subseteq$-extension. Whenever U is a non-normal κ-complete over κ we will denote by $\mathbb{T}_U$ the *Tree-Prikry forcing* relative to U (see [Git10, §1]).

2. Non-normal coherent sequences

Let us fix κ a measurable cardinal. Given two κ-complete (non-trivial) ultrafilters U, W over κ we shall say that U *is Mitchell below* W and write $U \lhd W$ whenever $U \in \mathrm{Ult}(V,W)$. Certainly, this is the natural generalization of the classical *Mitchell order* $\lhd$ between normal measures [Mit10].

We define two types of coherent sequences of ultrafilters; namely, *generalized coherent sequences* (Definition 2.1) and *almost coherent sequences* (Definition 2.5).

Definition 2.1. A sequence

$$\vec{U} = \langle U(\alpha,i) \mid \alpha<\kappa, i< o^{\vec{U}}(\alpha)\rangle^\frown\langle U(\kappa,i)\mid i<\gamma\rangle$$

is a *generalized coherent sequence of length* γ *with a top cardinal* κ if:

(i) There is a function $\pi:\kappa\to\kappa$ such that for every $i<\gamma$ $[\pi]_{U(\kappa,i)}=\kappa$.

(ii) For each $\alpha<\kappa$ and $\beta< o^{\vec{U}}(\alpha)$, $U(\alpha,\beta)$ is a $\pi(\alpha)$-complete ultrafilter over $\pi(\alpha)$. Also for $i<\gamma$, $U(\kappa,i)$ is a κ-complete ultrafilter over κ.

(iii) For each $\alpha<\kappa$ and $i<o^{\vec{U}}(\alpha)$, $[\pi\restriction\pi(\alpha)]_{U(\alpha,i)}=\pi(\alpha)$.

(iv) For every $\alpha\le\kappa$ and $\beta<o^{\vec{U}}(\alpha)$,

$$j_{U(\alpha,\beta)}(\vec{U})([\mathrm{id}]_{U(\alpha,\beta)}) = \langle U(\alpha,i)\mid i<\beta\rangle,$$

where $\vec{W}(\gamma)$ denotes the values of the sequence $\vec{W}$ at γ; i.e.,

$$\langle W(\gamma,i)\mid i<o^{\vec{W}}(\gamma)\rangle.$$

We say that $\vec{U}$ is *special*, if

(v) whenever $U(\alpha,i)$ is non-normal, $j_{U(\alpha,i)}(\vec{U})(\pi(\alpha))=\langle\rangle$.

Remark 2.2. If all the measures in $\vec{U}$ are normal then one recovers the standard notion of a coherent sequence of measures [Mit10].

Theorem 2.3. *Assume the* GCH *holds. Suppose that* $\langle U_i \mid i < \gamma\rangle$ *(with* $\gamma < \kappa$*) is a* $\triangleleft$*-increasing sequence of* κ*-complete ultrafilters over* κ *(which are not necessarily normal). Then there is a generalized coherent sequence* $\vec{U}$ *of length* γ *with a top cardinal* κ *such that* $U(\kappa, i) = \gamma$ *for every* $i < \gamma$*. Moreover if*

$$\{U_i \mid i < \gamma\} \cup \{U_i^{nor} \mid U_i \textit{ is not normal}\}$$

are distinct ultrafilters, then we can ensure that the sequence is special.

Proof. Start by finding a sequence of sets

$$\mathcal{A} = \{A_\alpha \mid \alpha < \gamma\} \cup \{A'_\alpha \mid U_\alpha \text{ is non-normal}\}$$

such that:

(1) For all $\alpha < \gamma$, $A_\alpha \in U_\alpha$ and $\min(A_\alpha) > \gamma$.
(2) If $A, B \in \mathcal{A}$ and $A \neq B$ then $A \cap B = \emptyset$.
(3) If U_α is non-normal then $A'_\alpha \in U_\alpha^{\text{nor}}$.

Such a sequence exists if

$$\{U_i \mid i < \gamma\} \cup \{U_i^{\text{nor}} \mid U_i \text{ is not normal}\}$$

is a set of less than κ-many distinct κ-complete ultrafilter. Otherwise, we just require that $\mathcal{A} = \{A_\alpha \mid \alpha < \gamma\}$ and ignore (3). Next, find $\pi : \kappa \to \kappa$ such that for every $i < \gamma$, $[\pi]_{U_i} = \kappa$, and define by induction on $\alpha < \gamma$, $\vec{V}^{(\alpha)}$, such that:

(1) $\text{dom}(V^{(0)}) = \{\langle\kappa, 0\rangle\}$ and $V^{(0)}(\kappa, 0) = U_0$.
(2) $\vec{V}^{(\alpha)}$ is a generalized coherent sequence of length $\alpha + 1$ with a top cardinal κ.
(3) $\alpha < \beta < \omega_1 \Rightarrow \vec{V}^{(\alpha)} \subseteq \vec{V}^{(\vec{\beta})}$ (as partial functions).
(4) For $\alpha > 0$, $\text{dom}(\vec{V}^{(\alpha)}) \setminus \bigcup_{\beta<\alpha} \text{dom}(V^{(\vec{\beta})}) = B_\alpha \times \alpha \cup \{\langle\kappa, \alpha\rangle\}$, where $B_\alpha \subseteq A_\alpha$ and $B_\alpha \in U_\alpha$.
(5) $V^{(\alpha)}(\kappa, \alpha) = U_\alpha$.
(6) For every $(\eta, i) \in \text{dom}(\vec{V}^{(\alpha)})$, $B_i \cap \pi(\eta) \in V^{(\alpha)}(\eta, i)$

In the moreover case we also require that:

(7) $A'_i \cap \pi(\eta) \in \pi^\eta_*(V^{(\alpha)}(\eta, i))$ for all i such that $V^{(\alpha)}(\eta, i)$ is non-normal and where $\pi^\eta = \pi \restriction \eta$.

The following claim says that it suffices to construct the sequence above.

Claim 2.4. *Let* $(\vec{V}^{(\alpha)})_{\alpha<\gamma}$ *be a sequence satisfying* $(1) - (6)$ *as above and let* $\beta \leq \gamma$*, then* $\vec{V} = \bigcup_{\alpha<\beta} \vec{V}^{(\alpha)}$ *is a coherent sequence of length* β *with a top cardinal* κ*. Moreover, if* (7) *holds than the sequence is special.*

Proof of claim. By (1),(3),(4), $\text{dom}(\vec{V}) = (\bigcup_{0<\beta<\alpha} B_\beta \times \beta) \cup \{\kappa\} \times \alpha$ and $\vec{V} \restriction \text{dom}(\vec{V}^{(\alpha)}) = \vec{V}^{(\alpha)}$. Hence (i)-(iii) are trivial.

To see (iv), let $(\eta,\beta)\in\mathrm{dom}(\vec{V})$ and there is $\alpha<\gamma$ such that $(\eta,\beta)\in\mathrm{dom}(V^{(\alpha)})$, then $\beta\leq\alpha$ and $V(\eta',i)=V^{(\alpha)}(\eta',i)$ for every $\eta'\in B_{\beta'}$ and $i<\beta'$, for some $\beta'\leq\beta$. Hence for every $\rho\in B_\beta\cap\pi(\eta)$, $\vec{V}(\rho)=\vec{V}^{(\alpha)}(\rho)$. By (6), $B_\beta\cap\pi(\eta)\in V(\eta,\beta)$ and therefore $j_{V(\eta,\beta)}(\vec{V})([id]_{V(\eta,\beta)})=j_{V(\eta,\beta)}(\vec{V}^{(\alpha)})([id]_{V(\eta,\beta)})$. Using the coherency of $\vec{V}^{(\alpha)}$,

$$j_{V(\eta,\beta)}(\vec{V})([id]_{V(\eta,\beta)})=j_{V^{(\alpha)}(\eta,\beta)}(\vec{V}^{(\alpha)})([id]_{V^{(\alpha)}(\eta,\beta)})=$$

$$=\langle V^{(\alpha)}(\eta,i)\mid i<\beta\rangle=\langle V(\eta,i)\mid i<\beta\rangle.$$

Finally, to see (v), we note that if $V(\eta,\beta)$ is non-normal, then by (7) $A'_\beta\cap\pi(\eta)\in\pi^\eta_*(V^{(\alpha)}(\eta,\beta))$ and since this set is disjoint from the domain of $\vec{V}^{(\alpha')}$ for every $\alpha'<\gamma$, it is disjoint from $\mathrm{dom}(\vec{V})$ and therefore $\pi(\eta)\notin\mathrm{dom}(j_{V(\eta,\beta)}(\vec{V}))$. We conclude that $j_{V(\eta,\beta)}(\vec{V})(\pi(\eta))=\langle\rangle$. □

Let us turn to the inductive definition of $\vec{V}^{(\alpha)}$, let $\mathrm{dom}(\vec{V}^{(0)})=\{\langle\kappa,0\rangle\}$ and $\vec{V}^{(0)}(\kappa,0)=U_0$. Now suppose that $V^{(\beta)}$ has been defined for $\beta<\alpha$. By the previous claim, letting $\vec{V}=\bigcup_{\beta<\alpha}\vec{V}^{(\beta)}$, we have that $\vec{V}$ is a generalized coherent sequence of length α with a top cardinal κ and $\mathrm{dom}(\vec{V})=(\bigcup_{0<\beta<\alpha}B_\beta\times\beta)\cup\{\kappa\}\times\alpha$. By (2), we let $\vec{V}^{(\alpha)}\restriction\mathrm{dom}(\vec{V})=\vec{V}$, and by (5), we have to define $V^{(\alpha)}(\kappa,\alpha)=U_\alpha$. By (4), it remains to define $B_\alpha\subseteq A_\alpha$ and $V^{(\alpha)}\restriction B_\alpha\times\alpha$. Towards this, since $\alpha<\kappa$ and since we started with a Mitchell increasing sequence of ultrafilters, we have $\langle U_i\mid i<\alpha\rangle\in M_{U_\alpha}$, hence we can find a function such that $\langle U_i\mid i<\alpha\rangle=[\eta\mapsto\langle V^\eta_i\mid i<\alpha\rangle]_{V_\alpha}$. Also,

(a) $M_{U_\alpha}\models\vec{V}=(j_{U_\alpha}(\vec{V})\restriction\kappa)^\frown\langle U_i\mid i<\alpha\rangle$ is coherent.
(b) For $i<\alpha$, $M_{U_\alpha}\models j_{U_\alpha}(A_i)\cap\kappa=A_i\in U_i$.
(c) $M_{U_\alpha}\models$ if U_i is non-normal then $j_{U_\alpha}(A'_i)\cap\kappa=A'_i\in\pi_*(U_i)=(j_{U_\alpha}(\pi)\restriction\kappa)_*(U_i)$.
(d) For $i<\alpha$, $M_{U_\alpha}\models[j_{U_\alpha}(\pi)\restriction\kappa]_{U_i}=[\pi]_{U_i}=\kappa=j_{U_\alpha}(\pi)(\kappa)$.

Reflecting this, we can find a set $B_\alpha\in V_\alpha$ such that for every $\eta\in B_\alpha$,

(a) $\vec{V}\restriction\pi(\eta)^\frown\langle V^\eta_0,...,V^\eta_\alpha\rangle$ is coherent with a top cardinal $\pi(\eta)$.
(b) For $i<\alpha$, $A_i\cap\pi(\eta)\in V^\eta_i$.
(c) $A_i\cap\pi(\eta)\in\pi^\eta_*(V^\eta_i)$ if V^η_i is non-normal.
(d) For $i<\alpha$, $[\pi\restriction\pi(\eta)]_{V^\eta_i}=\pi(\eta)$.

For $\eta\in B_\alpha$ and $i<\alpha$, let $V^{(\alpha)}(\eta,i)=V^\eta_i$.

Let us check $(1)-(7)$. First $(1),(3),(4),(5)$ are trivial. Condition $(6),(7)$ follows from the induction hypothesis and conditions $(b),(c)$ above. It remains to check (2), i.e. that $V^{(\alpha)}$ is a generalized coherent sequence: (i)-(iii) follows directly from the construction.

To see (iv), let $(\eta,\beta)\in\mathrm{dom}(\vec{V}^{(\alpha)})$. If $(\eta,\beta)\in\mathrm{dom}(\vec{V})$, then $V^{(\alpha)}(\eta,\beta)=V(\eta,\beta)$. Note that $B_\alpha\cap\pi(\eta)\notin V^{(\alpha)}(\eta,\beta)$ (which are the only cardinals where we made changes below $\pi(\eta)$ in $\vec{V}^{(\alpha)}$) and thus

$$j_{V(\eta,\beta)}(\vec{V}^{(\alpha)})([id]_{V(\eta,\beta)}) = j_{V(\eta,\beta)}(\vec{V})([id]_{V(\eta,\beta)}).$$

By the induction hypothesis, we have that

$$j_{V(\eta,\beta)}(\vec{V})([id]_{V(\eta,\beta)}) = \langle V(\eta,i) \mid i<\beta\rangle = \langle V^{(\alpha)}(\eta,i) \mid i<\beta\rangle,$$

and so we are done. If $(\eta,\beta) \in \mathrm{dom}(\vec{V}^{(\alpha)}) \setminus \mathrm{dom}(\vec{V})$, then either $\eta \ni B_\alpha$, in which case, by (a), $\vec{V} \restriction \eta^\frown\langle V_i^\eta \mid i<\alpha\rangle$ is coherent. Again, $\vec{V}^{(\alpha)} \restriction \eta$ defers from $\vec{V}^{(\alpha)} \restriction \eta$ only on $B_\alpha \cap \eta$ which is measure 0 with respect to $V(\eta,\beta)$, for every $i<\alpha$. Hence the ultrapower by $V(\eta,\beta)$ will still satisfy the coherency requirement in (iv). The case $\eta=\kappa$ is similar.

Finally to see (v), use (c) and note that for every (η,β) for which $V^{(\alpha)}(\eta,\beta)$ is non-normal, $A'_\beta \cap \pi(\eta) \in \pi_*^\eta(V^{(\alpha)}(\eta,\beta))$. This set is disjoint from the domain of $\vec{V}^{(\alpha)}$ and therefore

$$j_{V^{(\alpha)}(\eta,\beta)}(\vec{V}^{(\alpha)})(\pi(\alpha)) = \emptyset. \qquad \square$$

We will need also a particular case of a generalized coherent sequence which we call *almost coherent sequence*:

Definition 2.5. An *almost coherent sequence* is a sequence

$$\vec{U} = \langle U(\alpha,\beta) \mid \alpha \le \kappa, \beta < o^{\vec{U}}(\alpha)\rangle$$

such that:

(1) $U(\alpha,0)$ is an α-complete (non-necessarily normal) ultrafilter over α.
(2) for $\alpha\le\kappa$ and $0<\beta<o^{\vec{U}}(\alpha)$, $U(\alpha,\beta)$ is a normal measure on α
(3) for every $\langle\alpha,\beta\rangle \in \mathrm{dom}(\vec{U})$,

$$j_{U(\alpha,\beta)}(\vec{U}) \restriction \alpha+1 = \vec{U} \restriction (\alpha,\beta)$$

where

$$\vec{W} \restriction \alpha+1 = \langle W(\gamma,\beta) \mid \gamma\le\alpha,\ \beta<o^{\vec{W}}(\gamma)\rangle$$

and

$$\vec{W} \restriction (\alpha,\beta) = \vec{W} \restriction \alpha^\frown\langle W(\alpha,\gamma) \mid \gamma<\beta\rangle$$

whenever $\vec{W}$ is an almost coherent sequence

In the above definition if $\beta=0$ then $j_{U(\alpha,0)}(\vec{U}) \restriction \alpha+1 = \vec{U} \restriction \alpha$. That is, we require that there are no measures on α in $j_{U(\alpha,0)}(\vec{U})$.

Corollary 2.6. *Let $\langle V_\alpha \mid \alpha<\omega_1\rangle$ a $\triangleleft$-increasing sequence be such that V_α is normal for all $\alpha>0$. There is an almost coherent sequence $\vec{U}$ such that $\vec{U}(\kappa,\alpha)=V_\alpha$ for all $\alpha<\omega_1$.*

Note that $\{V_0^{\mathrm{nor}}\} \cup \{V_i \mid i<\gamma\}$ are all distinct ultrafilters, and therefore we can make the coherent sequence special. The proof of the following proposition is a straightforward verification:

Proposition 2.7. *Suppose that $\vec{U}$ is a generalized coherent sequence of length γ with a top cardinal κ, then for each $\alpha \leq \kappa$ and $i \leq o^{\vec{U}}(\alpha)$, $\langle U(\beta, r) \mid \beta < \alpha, r < \min(o^{\vec{U}}(\beta), i)\rangle^\frown\langle U(\alpha, j) \mid j < i\rangle$ is a generalized coherent sequence of length i with a top cardinal α.*

We denote the above generalized coherent sequence by $\vec{U} \restriction (\alpha, i)$.

3. Non-normal Magidor forcing with a coherent sequence

In this section we generalize the presentation of Magidor forcing due to Mitchell [Mit82] (see also [Git10]) which has been also studied by the first author and Gitik in a series of papers [BG21, BG22a, BG22b].

Proposition 3.1. *Let $\vec{U}$ be a generalized coherent sequence with a top cardinal κ and let $\langle A_i \mid i < o^{\vec{U}}(\kappa)\rangle$ be a sequence of sets such that $A_i \in U(\kappa, i)$. Then for every $i < \kappa$,*

$$\{\nu \in A_i \mid o^{\vec{U}}(\nu) = i,\ \forall j < i\ A_j \cap \pi(\nu) \in U(\nu, j)\} \in U(\kappa, i).$$

Proof. For every $i < o^{\vec{U}}(\kappa)$, and $j < i$,

$$j_{U(\kappa,i)}(A_j) \cap j_{U(\kappa,i)}(\pi)([\mathrm{id}]_{U(\kappa,i)}) = A_j$$

and by coherency, $U(\kappa, j) = j_{U(\kappa,i)}(\vec{U})([\mathrm{id}]_{U(\kappa,i)}, j)$. It follows that

$$M_{U(\kappa,i)} \models \forall j < i,\ j_{U(\kappa,i)}(A_j) \cap j_{U(\kappa,i)}(\pi)([\mathrm{id}]_{U(\kappa,i)}) \in j_{U(\kappa,i)}(\vec{U})([\mathrm{id}]_{U(\kappa,i)}, j)$$

□

Notation 3.2. A *basic pair* is a pair (α, A) where

$$A \in \bigcap_{i<o^{\vec{U}}(\alpha)} U(\alpha, i) =: \bigcap \vec{U}(\alpha).$$

By convention, if $o^{\vec{U}}(\alpha) = 0$, then the sequence is empty, so universal statements about it will be vacuously true. For a basic pair $t = (\alpha, A)$, we denote $\kappa(t) := \alpha$ and $A(t) := A$.

Definition 3.3. Let $\vec{U}$ be a generalized coherent sequence of length γ with a top cardinal κ. We define the condition of the *non-normal-Magidor forcing* $\mathbb{M}[\vec{U}]$ as the poset consisting of conditions $\langle t_1, ..., t_n, \langle \kappa, A\rangle\rangle$ such that:

(1) each t_i is a basic pair.
(2) For each $\alpha \in A(t_i) \cup \{\kappa(t_i)\},\ \pi(\alpha) > \kappa(t_{i-1})$.

Definition 3.4. The order for $\mathbb{M}[\vec{U}]$ is defined by

$$\langle t_1, ..., t_n, \langle \kappa, A\rangle\rangle \leq \langle s_1, ..., s_m, \langle \kappa, B\rangle\rangle$$

whenever there are indices $i_0 := 0 < 1 \leq i_1 < ... < i_n \leq m =: i_{n+1}$ such that for each $1 \leq r \leq n+1$

(1) $\kappa(s_{i_r}) = \kappa(t_r)$, $A(s_{i_r}) \subseteq A(t_r) \setminus \pi^{-1}[\kappa(s_{i_{r-1}}) + 1]$.
(2) If $i_{r-1} < j < i_r$, then
    (a) $\kappa(s_j) \in A(t_r)$.

(b) $o^{\vec{U}}(\kappa(s_j)) < o^{\vec{U}}(\kappa(t_r))$.
(c) $A(s_j) \subseteq (A(t_r) \cap \pi(\kappa(s_j))) \setminus \pi^{-1}[\kappa(s_{j-1}) + 1]$.

In case $n = m$ (and therefore $i_r = r$) we write $p \leq^* q$.

Proposition 3.5. *The order $\leq$ on $\mathbb{M}[\vec{U}]$ is transitive*

Proof. Suppose

$$\langle t_1, ..., t_n, \langle \kappa, A \rangle\rangle \leq \langle s_1, ..., s_m, \langle \kappa, B \rangle\rangle \leq \langle z_1, ..., z_k, \langle \kappa, C \rangle\rangle.$$

By definition there are

$$1 \leq i_1 < ... \leq i_n \leq m \text{ and } 1 \leq j_1 < j_2 < ... < j_m \leq k$$

witnessing the left and right inequalities (resp.). Define $l_r = j_{i_r}$. then $1 \leq l_1 < ... < l_n \leq k$. Let us prove that $(1), (2a) - (2c)$ hold:

(1) First, $\kappa(z_{l_r}) = \kappa(z_{j_{i_r}}) = \kappa(s_{i_r}) = \kappa(t_r)$. Moreover,

$$A(z_{l_r}) \subseteq A(s_{i_r}) \cap \pi^{-1}[\kappa(z_{l_r - 1}) + 1] \subseteq A(t_r) \setminus \pi^{-1}[\kappa(z_{l_r - 1}) + 1].$$

(2) Suppose that $l_{r-1} < j < l_r$. and let us split into cases:

Case 1: There is $1 \leq w \leq m$ such that $j = j_w$, in which case, $i_{r-1} < w < i_r$ and therefore
(a) $\kappa(z_j) = \kappa(s_w) \in A(t_r)$.
(b) $o^{\vec{U}}(\kappa(z_j)) = o^{\vec{U}}(\kappa(s_w)) < o^{\vec{U}}(\kappa(t_r))$.
(c) $A(z_j) \subseteq A(s_w) \setminus \pi^{-1}[\kappa(z_{j-1}) + 1] \subseteq$
$\subseteq A(t_r) \cap \pi(\kappa(z_j)) \setminus \pi^{-1}[\kappa(z_{j-1}) + 1]$.

Case 2. $j_{w-1} < j < j_w$, in which case, $i_{r-1} < w \leq i_r$ and therefore
(a) $\kappa(z_j) \in A(s_w) \subseteq A(t_r)$.
(b) $o^{\vec{U}}(\kappa(z_j)) < o^{\vec{U}}(\kappa(s_w)) \leq o^{\vec{U}}(\kappa(t_r))$.
(c) $A(z_j) \subseteq A(s_w) \cap \pi(\kappa(z_j)) \setminus \pi^{-1}[\kappa(z_{j-1}) + 1] \subseteq$
$\subseteq A(t_r) \cap \pi(\kappa(z_j)) \setminus \pi^{-1}[\kappa(z_{j-1}) + 1]$.

□

Remark 3.6. By condition $(2b)$ of the order on $\mathbb{M}[\vec{U}]$, given a set $A \in \cap\vec{U}(\alpha)$, we may assume always that $A = \biguplus_{j < o^{\vec{U}}(\alpha)} A^{(j)}$, where

$$A^{(j)} = \{\nu \in A \mid o^{\vec{U}}(\nu) = j\}.$$

Notation 3.7. Given $p = \langle t_1, ..., t_n, \langle \kappa, A \rangle\rangle \in \mathbb{M}[\vec{U}]$, let

(1) $l(p) = n$.
(2) $t_i(p) = t_i$,
(3) $t_{n+1}(p) = \langle \kappa, A \rangle$ and in particular $A(t_{n+1}(p)) = A$.
(4) $p \restriction i + 1 = \langle t_1, ..., t_i \rangle$.
(5) $p \restriction [i + 1, n + 1] = \langle t_{i+1}, ..., t_n, t_{n+1} \rangle$.

The following are straightforward:

Proposition 3.8. *$\mathbb{M}[\vec{U}]$ is κ-centered and therefore κ^+-cc.*

Proposition 3.9. *For $p \in \mathbb{M}[\vec{U}]$, $(\mathbb{M}[\vec{U}]/p, \leq^*)$ is $\kappa(t_1(p))$-directed-closed.*

Proposition 3.10. *Given* $p \in \mathbb{M}[\vec{U}]$ *and* $1 \leq i \leq l(p)$

$$\mathbb{M}[\vec{U}]/p \simeq \big(\mathbb{M}[\vec{U} \restriction \pi(t_i(p))]/p \restriction i+1\big) \times \big(\mathbb{M}[\vec{U}]/p \restriction [i+1, n+1]\big)$$

Definition 3.11. Let $p = \langle t_1, .., t_n, t_{n+1}\rangle \in \mathbb{M}[\vec{U}]$ and $\alpha \in A(t_i)$, for some $1 \leq i \leq n+1$, define

$p^\frown\langle\alpha\rangle = \langle t_1, ..., t_{i-1}, \langle \alpha, A(t_i) \cap \pi(\alpha)\rangle, \langle \kappa(t_i), A(t_i) \setminus \pi^{-1}[\alpha+1]\rangle, t_{i+1}, ..., t_{n+1}\rangle$.

We define recursively $p^\frown\langle\alpha_1, .., \alpha_n\rangle = (p^\frown\langle\alpha_1, ..., \alpha_{n-1}\rangle)^\frown\langle\alpha_n\rangle$.

The proof for the Prikry property and the strong Prikry property will be given in the next section for the non-normal Radin forcing, but the proof is completely analogous and therefore is omitted.

Definition 3.12. A tree $T \subseteq [\kappa]^{<\omega}$ of height n, consisting of π-increasing sequence is called $\vec{U}$-fat if for every $t \in T$, such that $|t| < n$, there is $i < o^{\vec{U}}(\kappa)$ such that $\mathrm{Succ}_T(t) = \{\alpha \mid t^\frown\alpha \in T\} \in U(\kappa, i)$. Suppose that $\vec{V} = \langle \vec{v}_1, ..., \vec{v}_n\rangle$ is a sequence of generalized coherent sequences, a sequence of trees $\vec{T} = \langle T_1, ..., T_n\rangle$ is called $\vec{v}$-fat if for each $1 \leq i \leq n$, T_i is $\vec{v}_i$-fat.

If for every $1 \leq i \leq n$, the coherent sequences $\vec{v}_i$ above happens to be the coherent sequence $\vec{U} \restriction \pi(\kappa(t_i(p)))$ for some given condition p of length n, then, we say that T is fat below p. For a tree T of height n, we denote the set of maximal branches in T by $mb(T) = \{t \in T \mid |t| = n\}$.

Theorem 3.13 (The strong Prikry property)**.** *Let* $p \in \mathbb{M}[\vec{U}]$ *be any condition and* $D \subseteq \mathbb{M}[\vec{U}]$ *be dense open. Then there* $p \leq^* p^*$ *and and a sequence* $\vec{T} = \langle T_1, ..., T_{l(p)+1}\rangle$ *fat below* p^* *such that for every sequence of branches* $\langle b_1, ..., b_{l(p)+1}\rangle \in \prod_{1 \leq i \leq l(p)+1} mb(T_i)$,

$$p^{*\frown} b_1^\frown b_2 ...^\frown b_n \in D$$

Corollary 3.14. $\mathbb{M}[\vec{U}]$ *preserves all cardinals.*

Other properties of the classical Magidor forcing $\mathbb{M}[\vec{U}]$ can be generalized to our non-normal version – this shall not be presented here. For more about these properties see [BG21, BG22a, BG22b].

One important difference between this forcing and the usual normal Magidor forcing is that the generic sequence is not closed anymore:

Definition 3.15. Let $G \subseteq \mathbb{M}[\vec{U}]$ be V-generic. The generic object added by G is

$$C_G := \{\alpha \mid \exists p \in G \ \exists 1 \leq i \leq l(p),\ \kappa(t_i(p)) = \alpha\}.$$

It is not hard to check that for every $A \in \bigcap \vec{U}(\kappa)$, $C_G \subseteq^* A$ and that $V[G] = V[C_G]$. However, this sequence is not normal:

Proposition 3.16. *Let* $\vec{U}$ *be a generalized coherent sequence coherent sequence of length* γ *with a top cardinal* κ*, and let* G *be generic for* $\mathbb{M}[\vec{U}]$*. For* $0 < \alpha < \gamma$*,* $U(\kappa, \alpha)$ *is normal if and only if there is* $\xi < \kappa$ *such that for every* $\rho \in C_G \setminus \xi$ *with* $o^{\vec{U}}(\rho) = \alpha$*,* $\sup(C_G \cap \rho) = \rho$*.*

Proof. Suppose that $U(\kappa,\alpha)$ is normal, then $[\pi]_{U(\kappa,\alpha)} = [id]_{U(\kappa,\alpha)}$. Hence there is $A \in \cap\vec{U}(\kappa)$ such that for every $\rho \in A$, if $o^{\vec{U}}(\rho) = \alpha$ then $\pi(\rho) = \rho$. Hence there is $\xi < \kappa$ such that $C_G \setminus \xi \subseteq A$. Now suppose that $\rho \notin C_G \setminus \xi$ and $o^{\vec{U}}(\rho) = \alpha$ and let $p \in G$ be a condition such that $\kappa(t_i(p)) = \rho$. The for every $\delta' < \pi(\rho) = \rho$, and for every $p \leq q$, there is j such that $\kappa(t_j(q)) = \rho$ and therefore there is $\gamma \in A(t_j(q)) \setminus \delta'$. Now $q^\frown\langle\delta'\rangle$ is a condition forcing that $\sup(C_G \cap \rho) \geq \delta'$. By density, there is such a condition in G and since $\delta' < \rho$ was arbitrary, $\sup(C_G \cap \rho) = \rho$. If $U(\kappa,\alpha)$ is non-normal, then there is $\xi < \kappa$ such that for every $\rho \in C_G \setminus \xi$ with $o^{\vec{U}}(\rho) = \alpha$, $\pi(\rho) < \rho$. Now let $p \in C_G$ be any condition with $\rho = \kappa(t_i(p))$ for some $1 \leq i \leq p$, then for every $j > i$, and every $\alpha \in A(t_j(p))$, $\pi(\alpha) > \kappa(t_{j-1}) \geq \rho$, $p \Vdash C_G \cap \rho = C_G \cap \pi(\rho)$. Thus $\sup(C_G \cap \rho) \leq \pi(\rho) < \rho$. □

Lemma 3.17. *Let α be a regular cardinal in V. If $p \Vdash \alpha \notin \operatorname{acc}(C_G)$, then $p \Vdash \operatorname{cf}(\alpha) = \alpha$. In particular all cofinalities below $\delta_0 := \min\{\nu \mid o^{\vec{U}}(\nu) > 0\}$ are preserved.*

The non-normal Magidor-Radin forcing appeared implicitly in the work of Merimovich [Mer11, Mer03a]. The analogy is the following: the tree-Prikry forcing appears as a projection of both the usual Gitik-Magidor Extender-Based Prikry forcing of [GM94] and its more modern presentation due to Merimovich [Mer03b]. Next, we will show that the non-normal Magidor forcing $\mathbb{M}[\vec{U}]$ appears as a projection of Merimovich's Extender-Based Magidor/Radin forcing [Mer11].

Given a Mitchell increasing sequence of (short) extenders

$$\bar{E} = \langle E_\xi \mid \xi < \gamma\rangle$$

a condition in the forcing $\mathbb{P}_{\bar{E}}$ has the form

$$\langle\langle f_0, A_0, \bar{e}_0\rangle, ..., \langle f_n, A_n, \bar{e}_n\rangle, \langle f, A, \bar{E}\rangle\rangle,$$

where

(1) $\bar{e}^i = \langle\bar{e}^i_j \mid j < o(\bar{e}^i)\rangle$ is an extender sequence with critical point κ_i.
(2) $\operatorname{dom}(f_i) \in P_{\kappa_i^+}(\mathfrak{D}_i)$, $f_i : \operatorname{dom}(f_i) \to \mathfrak{R}_i^{<\omega}$, where $\mathfrak{D}_i$ is the set of all possible coordinates for the extender sequence e^i and $\mathfrak{R}_i$ is the set of ranges for f_i (which consists of extender sequences[2] below $crit(e^i)$).
(3) A_i is a $\operatorname{dom}(f_i)$-tree; namely, for every $\vec{\nu} \in A_i$,

$$\operatorname{Succ}_{A_i}(\vec{\nu}) \in \bigcap_{j<o(e^i)} e^i_j(\operatorname{dom}(f_i)).$$

We refer the reader to Merimovich's paper [Mer11] for a complete account of the Magidor-Radin extender-based forcing.

[2]An extender sequence is a sequence of the form $\xi = \langle\rho\rangle^\frown\langle e_i \mid i < j\rangle$ where e_i is a Mitchell increasing sequence of extenders. where $crit(e_0) \leq \rho < j_{e_0}(\operatorname{crit}(e_0))$.We denote by $\xi_0 = \rho$ and $\bar{e}(\xi) = \langle e_i \mid i < j\rangle$.

Recall that if E is a (κ,λ)-extender and $\alpha<\lambda$, then $k_\alpha : M_{E_\alpha}\to M_E$ is an elementary embedding defined by $k_\alpha([f]_{E_\alpha})=j_E(f)(\alpha)$. The next proposition expresses that from a Mitchell-increasing sequence of extenders, one can derive many Mitchell-increasing sequences of ultrafilters.

Proposition 3.18. *Assume GCH. Suppose that E is a (κ,λ)-extender on κ, and $U\in M_E$ is an ultrafilter on κ such that $U=j_E(f)(\xi_1,...,\xi_n)$, then for any α with $\kappa,\xi_1,...,\xi_n\in rng(k_\alpha)$, $U\in M_{E_\alpha}$.*

Proof. By the assumption, we have that $\mathrm{crit}(k_\alpha)\geq(\kappa^{++})^{M_{E_\alpha}}$. Let $\rho_1,...,\rho_n$ be preimages of $\kappa,\xi_1,..,\xi_n$ under k_α respectively. We have

$$U'=j_{E_\alpha}(f)(\rho_1,...,\rho_n)\in M_{E_\alpha}.$$

Let us prove that $U'=U$. Indeed, for every $X\subseteq\kappa$ ($P(\kappa)$ is the same in all the models), we have that $X\in U'$ if and only if $k_\alpha(X)\in U$. But we have that $k_\alpha(X)=X$ as the critical point of k_α is above κ. □

Suppose that we are given for every $i<o(\bar{E})$, $\alpha_i<j_{E_0}(\kappa)$ such that $\langle E_i(\alpha_i)\mid \ell\leq i<o(\bar{E})\rangle$ is Mitchell increasing. Let $\vec{U}$ be the coherent sequence derived from $\langle E_i(\alpha_i)\mid \ell\leq i<o(\bar{E})\rangle$, and let us prove that $\mathbb{M}[\vec{U}]$ is a projection of $\mathbb{P}_{\bar{E}}$.

Theorem 3.19. *Let $\bar{E}$ be a Mitchell increasing sequence of extenders with $o(\bar{E})<\kappa=\mathrm{crit}(E)$. For every $\ell<o(\bar{E})$, let $\kappa\leq\alpha_\ell<j_{E_0}(\kappa)$ be an ordinal such that $\langle E_i(\alpha_i)\mid i<o(\bar{E})\rangle$ is $\triangleleft$-increasing and let $\vec{U}$ be the generalized coherent sequence derived from $\langle E_i(\alpha_i)\mid i<o(\bar{E})\rangle$. Then $\mathbb{M}[\vec{U}]$ is a projection of $\mathbb{P}_{\bar{E}}$ above a certain condition.*

Proof. Consider the condition $p_*=\langle\langle f_*,A_*,\bar{E}\rangle\rangle\in\mathbb{P}_{\bar{E}}$, where A_* is a d-tree such that $\mathrm{dom}(f_*)=\{\bar{\kappa}\}\cup\{\bar{\alpha}_i\mid i<o(\bar{E})\}$ and for each $\mathrm{dom}(f_*)$-object ν in A_*, and $i<o(\bar{E})$ with $o(\nu)=i$, $\mathrm{dom}(\nu)=\mathrm{dom}(f_*)$ and

$$(\star)\quad o(\nu(\bar{\alpha}_i))=o^{\vec{U}}(\nu(\bar{\alpha}_i)_0)\text{ and }\vec{U}(\nu(\bar{\alpha}_i)_0,j)=\bar{e}_j(\nu)(\nu(\bar{\alpha}_i)_0).$$

Note that such a set A_* exists since for every $i<o(\bar{E})$,

$$j_{E_i}(\vec{U})(\alpha_i)=k(j_{E_i(\bar{\alpha}_i)}(\vec{U})([id]_{E_i(\bar{\alpha}_i)}))$$

Since $\vec{U}$ is coherent and $U(\kappa,i)=E_i(\bar{\alpha}_i)$, we have that

$$j_{E_i(\bar{\alpha}_i)}(\vec{U})([id]_{E_i(\bar{\alpha}_i)})=\langle U(\kappa,j)\mid j<i\rangle=\langle E_j(\bar{\alpha}_j)\mid j<i\rangle$$

and since $i,\kappa^+<\mathrm{crit}(k)$, we conclude that

$$\langle U(\kappa,j)\mid j<i\rangle=k(\langle U(\kappa,j)\mid j<i\rangle)=j_{E_i}(\vec{U})(\alpha_i).$$

Finally, since $\kappa\leq\alpha_i<j_{E_0}(\kappa)$ we have

$$R_i(\bar{\alpha}_i)=\langle\alpha_i\rangle^\frown\langle E_j\mid j<i\rangle$$

(see Definition 4.4 in [Mer11]). Thus,

$$j_{E_i}(\vec{U})(mc_i(\bar{\alpha}_i)(j_{E_i}(\bar{\alpha}_i))_0)=j_{E_i}(\bar{U})(\alpha_i)=\langle U(\kappa,j)\mid j<i\rangle=$$

$$= \langle E_j(\bar{\alpha}_j) \mid j < i\rangle = \langle \bar{e}_j(mc_i(\bar{\alpha}_i)(j_{E_i}(\bar{\alpha}_i)))(\bar{\alpha}_j) \mid j < i\rangle.$$

Note that by squashing every $\nu \in A_*$ by some μ (i.e, taking $\nu \circ \mu^{-1}$), condition $(\star)$ does not changes.

Given a d-tree $B \subseteq Ob(d)^{<\omega}$, and $\bar{\gamma} = \langle \tau, e_0, e_1, \cdots e_\alpha \cdots \mid \alpha < o(\bar{\gamma})\rangle \in d$ (say $\kappa_0 = \text{crit}(e_0)$) we define

$$(B \restriction \bar{\gamma})^* = \{\nu(\bar{\gamma})_0 \mid \nu \in Ob(d),\ \forall \vec{\xi} \in B \cap V_{\nu(\kappa_0)}(\nu \in \text{Succ}_B(\vec{\xi}))\}.$$

Then

$$(B \restriction \bar{\gamma})^* \in \bigcap_{i<o(\bar{\gamma})} e_i(\bar{\gamma}).$$

Indeed, if $i < o(\bar{\gamma})$ and let $\vec{\xi} \in j_{e_i}(B) \cap V_{\kappa_0} = B$, then since B is a d-fat, $mc_i(d) \in j_{e_i}(\text{Succ}_B(\vec{\xi})) = \text{Succ}_{j_{e_i}(B)}(\vec{\xi})$. Thus, $mc_i(d)(j_{e_i}(\bar{\gamma}))_0 \in j_{e_i}(B \restriction \bar{\gamma})^*)$

Since every condition $p \in \mathbb{P}_{\bar{E}}/p_*$ is of the form $p \leq^* p_*^\frown\langle \nu_1, ..., \nu_k\rangle$, we can define $\pi(p)$ recursively on the length of p. First define $\pi(p_*) = \langle\langle \kappa, (A \restriction \overline{\alpha})^*\rangle\rangle$. Note that $(A \restriction \overline{\alpha})^* \in \bigcap_{i<o^{\vec{U}}(\kappa)} U(\kappa, i)$. Now given a condition

$$p = \langle\langle f^0, A_0, \bar{e}^0\rangle, ..., \langle f^n, A_n, \bar{e}^n\rangle, \langle f^{n+1}, A, \bar{E}\rangle\rangle, \in \mathbb{P}_{\bar{E}}/p_*,$$

suppose we have already defined

$$\pi(p) = \langle \vec{\alpha}_1, t_1, ..., \vec{\alpha}_n, t_n, \vec{\alpha}_{n+1}, t_{n+1}\rangle$$

such that for every $1 \leq i \leq n$,

(1) $\overline{\kappa(t_i)} \in \text{dom}(f^i) \cap (crit(\bar{e}^i), j_{\bar{e}^i_0}(crit(\bar{e}^i))$. We assume $\kappa(t_{n+1}) = \alpha$.
(2) $\vec{\alpha}_i = \langle f^i_k(\kappa(t_i))_0, ..., f^i_r(\kappa)_0\rangle$, where k is the minimal such that for every $l \geq k$, $o(f^i_l(\kappa(t_i))) = 0$. In particular $\vec{\alpha}_i$ is a sequence of ordinal of order 0,
(3) $A(t_i) = (A_i \restriction \overline{\kappa(t_i)})^*$.

Note that specifying $\kappa(t_i)$ completely determines $\pi(p)$ from p. In particular, any given $q \leq^* p$, has to be defined as a direct extension of $\pi(p)$ by shrinking for each $i \leq n+1$, $A(t_i) = (A_i \restriction \overline{\kappa(t_i)})^*$ to $(A^q_i \restriction \overline{\kappa(t_i)})^*$. In that case $\pi(q) \leq^* \pi(p)$, and for every direct extension $x \leq^* \pi(p)$ there is a direct extension $q \leq^* p$ such that $\pi(q) \leq^* x$.

Now given $\nu \in A_r$ for some $1 \leq r \leq n+1$,

<u>Case 1</u> if $o(\nu(\kappa(t_r)) = 0$, then for every $\bar{\gamma} \in \text{dom}(f^r)$, $o(\nu(\bar{\gamma})) = 0$ (see Definition 4.3 item (5) in [Mer11]) and therefore in $p^\frown\nu$ we only append points of order 0 to end extend the sequences of the Cohen function f^r (see Definition 4.4 in [Mer11]). So in this case

$$\pi(p^\frown\nu) = \langle \vec{\alpha}_1, t_1, ..., \vec{\alpha}'_r, t_r, ...\vec{\alpha}_{n+1}, t_{n+1}\rangle$$

where $\vec{\alpha}'_r = \vec{\alpha}_r^\frown \nu(\overline{\kappa(t_r)})_0$. Note that by definition, $\nu(\overline{\kappa(t_r)})_0 \in (A_r \restriction \overline{\kappa(r_t)})^* = A(t_i)$ and therefore $\pi(p^\frown\nu)$ is a legitimate one-point extension of $\pi(p)$.

Case 2: if $o(\nu(\overline{\kappa(t_r)})) > 0$, We define

$$\pi(p^\frown\nu) = \langle \vec{\beta}_1, s_1, ...\vec{\beta}_{n+1}, s_{n+1}, \vec{\beta}_{n+1}, s_{n+1}\rangle$$

by specifying

$$\kappa(s_i) = \begin{cases} \kappa(t_i) & i < r \\ \nu(\kappa(t_r))_0 & i = r \\ \kappa(t_{i-1}) & r < i \leq n+1 \end{cases}$$

By definition, $\nu(\overline{\kappa(t_r)})_0 \in (A_r \restriction \overline{\kappa(r_t)})^* = A(t_r)$, and therefore $\nu(\kappa(t_r))_0$ is a legitimate ordinal to be added to $\pi(p)$. Denote by $\bar{e} = \bar{e}(\nu)$ we note that by $(\star)$,

$$U(\nu(\kappa(t_r)), j) = \bar{e}_j(\overline{\kappa(t_r)}).$$

Therefore $A(s_r) = ((A \downarrow \nu) \restriction \overline{\nu(\kappa(t_r))})^* \in \bigcap_{j < o^{\vec{U}}(\nu(\kappa(r_t)))} U(\nu(\kappa(t_r)), j)$.

Also, note that every one-point extension $\pi(p)^\frown\rho$ of $\pi(p)$ using an order 0 ordinal there is some ν with the same order (this is due to $(\star)$) such that $\pi(p^\frown\nu) = \pi(p)^\frown\rho$. Since every extension of $\pi(p)$ is of the form $q \leq^* \pi(p)^\frown\langle\rho_1, .., \rho_n\rangle$, we conclude that π is a projection. □

4. Non-normal Radin forcing

Suppose that κ is a $\mathcal{P}_2\kappa$-hypermeasurable cardinal as witnessed by an elementary embedding $j : V \to M$. Let $\sigma < j(\kappa)$ and $\pi : \kappa \to \kappa$ be a function such that $j(\pi)(\sigma) = \kappa$. Following Cummings and Woodin's [Cum92b] we derive a sequence of ultrafilters as follows: $u^j(0) := \langle\sigma\rangle$ and for each $\xi \geq 1$

$$u^j(\xi) := \{X \subseteq V_\kappa \mid u^j \restriction \xi \in j(X)\}.$$

The construction of the u^j is continued until reaching ξ such that $u^j \restriction \xi \notin M$. The least ordinal ξ where the construction halts will be denoted $\ell(u^j)$.

Note that both $u^j(1)$ and $u^j(2)$ are κ-complete measure and that $u^j(2)$ concentrates on pairs $\langle\beta, w_\beta\rangle$ where w_β is a $\pi(\beta)$-complete ultrafilter over $V_{\pi(\beta)}$. Unlike in the usual construction of Radin forcing [Rad82] (see [Git10, §5.1]) our $u(1)$ here is a non-normal measure on V_κ.

Notation 4.1. For a sequence u as before σ_u denotes the ordinal in $u(0)$.

Definition 4.2 (Measure sequences and measure one sets)**.**

(1) $\mathcal{MS}_0 := \{u \in V_{\kappa+2} \mid \exists j \colon V \to M\ \exists\alpha \leq \ell(u^j)\ u = u^j \restriction \alpha\}$;
(2) $\mathcal{MS}_{n+1} := \{u \in \mathcal{MS}_n \mid \forall\xi \in [1, \ell(u))\ \mathcal{MS}_n \cap V_{j_u(\pi)(\sigma_u)} \in u(\xi)\}$.[3]

The collection of *measure sequences* $\mathcal{MS}$ is defined as $\bigcap_{n<\omega} \mathcal{MS}_n$.

Given $u \in \mathcal{MS}$ denote by $\mathscr{F}(u)$ the filter associate to u; namely,

$$\mathscr{F}(u) := \begin{cases} \{\emptyset\}, & \text{if } \ell(u) = 1; \\ \bigcap_{1 \leq \xi < \ell(u)} u(\xi), & \text{if } \ell(u) \geq 2. \end{cases}$$

[3]Here j_u stands for an embedding witnessing $u \in \mathcal{MS}_0$.

For $A \in \mathscr{F}(u)$ and $1 \le \xi < \ell(u)$ we will denote $(A)_\xi := \{w \in A \mid \ell(w) = \xi\}$.[4]

The next lemma due to Cummings [Cum92b] shows that one can derive long measure sequences from $\mathcal{P}_2\kappa$-hypermeasurable embeddings:

Lemma 4.3. *Let κ be a $\mathcal{P}_2\kappa$-hypermeasurable cardinal and $j\colon V \to M$ a witnessing embedding. Then, $\ell(u^j) \ge (2^\kappa)^+$ and $u^j \restriction \alpha \in \mathcal{MS}$ for $\alpha < \ell(u^j)$.*

In what follows u will be the truncation of the measure sequence u^j derived from j using some $\sigma < j(\kappa)$ as a seed; to wit, $u = u^j \restriction \alpha$ for some $\alpha < \ell(u^j)$. Likewise we will fix a function $\pi\colon \kappa \to \kappa$ such that $j(\pi)(\sigma) = \kappa$.

We define an ordering between measure sequences as follows:

Definition 4.4. Given $v, w \in \mathcal{MS} \cap V_\kappa$ write $v \prec w$ whenever $v \in V_{\kappa_w}$. Here we denoted $\kappa_w := \pi(\sigma_w)$.

Remark 4.5. Since $w \in V_\kappa$ it follows that $\sigma_w < \kappa$ and as a result $\pi(\sigma_w)$ is well-defined. Also, observe that $v \prec w$ entails $\sigma_u < \pi(\sigma_v)$.

We are now in conditions to define the non-normal Radin forcing:

Definition 4.6. The *Radin forcing* $\mathbb{R}_u$ consists of finite sequences

$$p = \langle \langle u_0^p, A_0^p\rangle, ..., \langle u^p_{\ell(p)-1}, A^p_{\ell(p)-1}\rangle, \langle u^p_{\ell(p)}, A^p_{\ell(p)}\rangle\rangle$$

where

(1) $u^p_{\ell(p)} = u$ and $u_i^p \in \mathcal{MS} \cap V_\kappa$ for all $i < \ell(p)$,
(2) $A_i^p \in \mathscr{F}(u_i^p)$ for all $i \le \ell(p)$,
(3) $\langle u_i^p \mid i < \ell(p)\rangle$ is $\prec$-increasing,
(4) and $u_i^p \prec v$ for all $v \in A^p_{i+1}$ and $i < \ell(p)$.

When p is clear from the context we will tend to suppress the superindex p.
Given $p, q \in \mathbb{R}_u$ write $p \le^* q$ whenever $\ell(p) = \ell(q)$, $u_i^p = u_i^q$ and $A_i^p \subseteq A_i^q$.

The minimal one-point extensions of a condition are given as follows:

Definition 4.7. Let $p = \langle\langle u_0, A_0\rangle, ..., \langle u_{n-1}, A_{n-1}\rangle, \langle u, A_n\rangle\rangle \in \mathbb{R}_u$, $i \le n$ and $v \in A_i$. We define the *one-point extension of p by v, $p^\frown v$*, as follows:

$$p^\frown v := \langle\langle u_0, A_0\rangle, ..., \langle v, A_i \downarrow v\rangle, \langle u_i, (A_i)_v\rangle, ..., \langle u_{n-1}, A_n\rangle, \langle u, A\rangle\rangle,$$

where $A_i \downarrow v := \{w \in A_i \cap V_{\kappa_v} \mid \ell(w) < \ell(v)\}$ and $(A_i)_v := \{w \in A_i \mid v \prec w\}$.
Given a (non-necessarily $\prec$-increasing) sequence $\langle v_i \mid i \le k\rangle$ one defines $p^\frown\langle v_i \mid i < k\rangle$ by recursion as $(p^\frown\langle v_i \mid i < k\rangle)^\frown v_k$.

For certain $v \in \mathcal{MS}$ it is plausible that $p^\frown v$ is not a well-defined condition. Let us call a condition $p \in \mathbb{R}_u$ *pruned* if $p^\frown\langle v_i \mid i \le k\rangle$ is a condition for all finite sequences $\langle v_i \mid i \le k\rangle$ in the measure one sets of p. It is not hard to show that the set of pruned conditions is $\le^*$-dense in $\mathbb{R}_u$. Thus, we do not loss any generality by assuming that all of our conditions in $\mathbb{R}_u$ are pruned.

[4]Note that $(A)_\xi \in \mathcal{F}(u \restriction \xi)$.

Definition 4.8 (The forcing ordering)**.** For two conditions $p, q \in \mathbb{R}_u$ we write $p \leq q$ if there is $\langle v_i \mid i \leq k\rangle \in \prod_{i\leq k} A^q_{j_i}$ such that $p \leq^* q^\frown\langle v_i \mid i \leq k\rangle$.

One can check that if $v, w \in A_i$ then $p^\frown\langle v, w\rangle = p^\frown\langle w, v\rangle$. This permits to show that $\leq$ is a transitive partial order relation on $\mathbb{R}_u$.

Remark 4.9. We point out that the map π representing κ can be used to establish a projection between $\mathbb{R}_u$ and the usual Radin forcing.

Lemma 4.10 (Some properties of $\mathbb{R}_u$)**.**

(1) $\mathbb{R}_u$ is a κ-centered poset;

(2) for each $p \in \mathbb{R}_u$ and $i < \ell(p)$,

$$\mathbb{R}_u/p \simeq (\mathbb{R}_{u_i}/p \restriction i+1) \times (\mathbb{R}_u/p \restriction [i+1, \ell(p));$$

(3) for each $p \in \mathbb{R}_u$ the poset $\langle \mathbb{R}_u/p, \leq^*\rangle$ is $\pi(\sigma_{u^p_0})$-directed-closed. □

Next we verify the *Strong Prikry property* for $\mathbb{R}_u$. For this we need the notion of a *fat tree* which, to our understanding, is due to Gitik [Git10, §5].

Definition 4.11 (Fat trees)**.** Let $n < \omega$ and $w \in \mathcal{MS}$. A tree $T \subseteq [\mathcal{MS} \cap V_{\kappa_w}]^{\leq n}$ consisting of $\prec$-increasing sequences is called *w-fat* if it is either the empty tree $\varnothing$ or for each $\langle v_0, \dots, v_k\rangle \in T$ with $k < n$,

$$\mathrm{Succ}_T(\langle v_0, \dots, v_k\rangle) \in w(\alpha) \text{ for some } \alpha < \ell(w).$$

Given a w-fat tree T we denote by $\mathcal{B}(T)$ the maximal branches of T.

Lemma 4.12 (Strong Prikry property)**.** *Let $p \in \mathbb{R}_u$ and $D \subseteq \mathbb{R}_u$ be a dense open set. There is $p \leq^* p^*$, $I \subseteq \ell(p)$ and $\mathcal{T} = \langle T_i \mid i \in I\rangle$ such that:*

(1) T_i is a u^p_i-fat tree.

(2) For each $\langle \vec{v}_i \mid i \in I\rangle$ with $\vec{v}_i \in \mathcal{B}(T_i)$, $p^{*\frown}\langle \vec{v}_0, \dots, \vec{v}_{\max(I)}\rangle \in D$.

Proof. Let p and D be as in the statement of the lemma. To streamline the argument let us assume that $p = \langle\langle u, A\rangle\rangle$. The general argument follows combining this base case with the factoring lemma (see Lemma 4.10).

For $q \in \mathbb{R}_u$ and $n \leq \ell(q)$ we denote by $L_n(q)$ the *nth-tail of* q; namely,

$$L_n(q) := q \restriction \ell(q) - n.$$

In the first part of the proof we define by induction a $\leq^*$-increasing sequence $\langle p^n \mid n < \omega\rangle$ of conditions with $p^0 := p$ such that for each $n \geq 1$ and each condition $p^n \leq q \in D$ with $\ell(q) \geq n$ the following hold:

(1) There is a u-fat tree T^q of height n with $L_n(q) \prec w$ for all $\langle w\rangle \in T^q$;

(2) For all $\vec{w} \in \mathcal{B}(T^q)$ there is $q_{\vec{w}} \in D$ such that $L_n(q)^\frown(p^{n\frown}\vec{w}) \leq^* q_{\vec{w}}$.

Clearly the above holds for any $q \in D$ such that $p^0 \leq q$ as witnessed by the tree $T^q := \varnothing$. Suppose by induction that $p^n = \langle(u, A^{p^n})\rangle$ has been defined.

Claim 4.13. *There is $p^n \leq^* p^{n+1}$ such that for each condition $p^{n+1} \leq q \in D$ with $\ell(q) \geq n+1$ Clauses (1) and (2) above hold.*

Proof of claim. Denote $\mathcal{L}_{n+1}(p^n) := \{L_{n+1}(q) \in V_\kappa \mid q \leq p^n,\, \ell(q) \geq n+1\}$.

For a fixed $L \in \mathcal{L}_{n+1}(p^n)$ denote by $A_0(L)$ the collection of all $v \in A^{p^n}$ for which there is a u-fat tree T^v of height n (with $v \prec w$ for all $\langle w \rangle \in T^v$) such that for each $\vec{w} \in \mathcal{B}(T^v)$, there is $q_{\vec{w},v} \in D$ with $L^\frown(p^{n\frown}\langle v, \vec{w}\rangle) \leq^* q_{\vec{w},v}$.

Denote $A_1(L) := A^{p^n} \setminus A_0(L)$. For each $\alpha < \ell(u)$ let $i_{\alpha,L} < 2$ be the unique index witnessing $A_{i_{\alpha,L}}(L) \in u(\alpha)$. Define $A^{n+1}_\alpha := \triangle_{L \in \mathcal{L}_{n+1}(p^n)} A_{i_{\alpha,L}}(L)$ and

$$A^{p^{n+1}} := (\textstyle\bigcup_{\alpha<\ell(u)} A^{n+1}_\alpha) \cap A^{p^n}.$$

We claim that $p^{n+1} := \langle\langle u, A^{p^{n+1}}\rangle\rangle$ is the sought condition. To show this let $p^{n+1} \leq q \in D$ be with $\ell(q) \geq n+1$. We shall find a u-fat tree T^q with height $n+1$ witnessing Clauses (1) and (2) with respect to p^{n+1}.

Since $p^n \leq q$ we can use our induction hypothesis to find a u-fat tree T^q of height n such that for all $\vec{w} \in \mathcal{B}(T^q)$ there is $q_{\vec{w}} \in D$ with

$$L_n(q)^\frown(p^{n\frown}\vec{w}) \leq^* q_{\vec{w}}.$$

In turn, $L_n(q)$ decomposes as $L_{n+1}(q)^\frown\langle v, B\rangle$ for some $v \in A^{p^{n+1}}$. Thus,

$$L_{n+1}(q)^\frown(p^{n\frown}\langle v, \vec{w}\rangle) \leq^* q_{\vec{w}}$$

and $v \in A^{n+1}_\alpha$ for some $\alpha < \ell(u)$. This means that

$$v \in A^{n+1}_\alpha \cap A_0(L_{n+1}(q))$$

and as a result $i_{\alpha, L_{n+1}(q)} = 0$. Thus, by definition, for each $v \in A^{n+1}_\alpha$ with $L_{n+1}(q) \prec v$ there is a u-fat tree T^v of height n (with $v \prec w$ for all $\langle w \rangle \in T^v$) such that for each $\vec{w} \in \mathcal{B}(T^v)$ there is $q_{\vec{w},v}$ with

$$L_{n+1}(q)^\frown(p^{n\frown}\langle v, \vec{w}\rangle) \leq^* q_{\vec{w},v} \in D.$$

Clearly $q_{\vec{w},v}$ is $\leq^*$-compatible with $L_{n+1}(q)^\frown(p^{n+1\frown}\langle v, \vec{w}\rangle)$. Thus it is harmless to assume that $q_{\vec{w},v}$ is in fact $\leq^*$-stronger than this latter condition. Thus it suffices to take $T^q := \{\langle v\rangle^\frown\vec{w} \mid v \in A^{n+1}_\alpha,\, L_{n+1}(q) \prec v,\, \vec{w} \in T^v\}$. $\square$

The above procedure defines a $\leq^*$-decreasing sequence $\langle p^n \mid n < \omega\rangle$ and we can let p^ω a $\leq^*$-lower bound for it. This condition allows us to get rid of the dependence on the lower parts. More formally: If $p^\omega \leq q \in D$ is a condition (say with $\ell(q) = n$) we can use the defining property of p^n to find a u-fat tree T^q of height n such that for all $\vec{w} \in \mathcal{B}(T^q)$ there is $q_{\vec{w}} \in D$ such that $p^{n\frown}\vec{w} \leq^* q_{\vec{w}}$ (here we have used that $L_n(q) := \emptyset$). Once again, since $q_{\vec{w}}$ and $p^{\omega\frown}\vec{w}$ are $\leq^*$-compatible we may assume that $p^{\omega\frown}\vec{w} \leq^* q_{\vec{w}}$.

Fix a condition $q \in D$ with $p^\omega \leq q$.

Claim 4.14. *There is $p^\omega \leq^* p^*$ and a u-fat tree S of length $\ell(q)$ such that $p^{*\frown}\vec{w} \in D$ for all $\vec{w} \in S$.*

Proof of claim. Let $p^\omega \leq q \in D$ and T be a u-fat tree of height $\ell + 1$ for which there is $q_{\vec{w}} \in D$ with $p^{\omega\frown}\vec{w} \leq^* q_{\vec{w}}$. Note that $q_{\vec{w}}$ takes the form

$$\langle (w_0, B_0(\vec{w})), \ldots, (w_\ell, B_\ell(\vec{w})), (u, B(\vec{w}))\rangle.$$

For each $j \leq \ell + 1$ let us denote

$$T \restriction j := \{\vec{v} \in [\mathcal{MS}]^{<\omega} \mid \vec{v} = \vec{w} \restriction j \text{ for some } \vec{w} \in T\}$$

and, for each $\vec{v} \in T \restriction j$, denote

$$T_{\vec{v}} := \{\vec{\mu} \in [\mathcal{MS}]^{<\omega} \mid \vec{v}^\frown \vec{\mu} \in T\}.$$

Fix $i \leq \ell$. For each $\vec{z} \in T \restriction (i+1)$ we shall be interested in the map

$$B_i(\vec{z}^\frown \langle \cdot \rangle) \colon T_{\vec{z}} \to \mathcal{F}(z_i)$$

given by $\vec{\mu} \mapsto B_i(\vec{z}^\frown \vec{\mu})$. Since all the measures involved in $T_{\vec{z}}$ are κ-complete we can find a u-fat tree $S(\vec{z}) \subseteq T_{\vec{z}}$ of height $(\ell+1) - i$ such that when restricting the above map to it becomes constant with value $B_i(\vec{z})$.

Let S_i denote the u-fat tree of height ℓ such that

- $(S_i) \restriction (i+1) = T \restriction (i+1)$;
- $(S_i)_{\vec{z}} = S(\vec{z})$ for all $\vec{z} \in T \restriction i+1$.

For each $\vec{v} \in T \restriction i$ there is $\alpha(\vec{v}) < \ell(u)$ such that $\mathrm{Succ}_T(\vec{v}) \in u(\alpha(\vec{v}))$ thus the set $B_i(\vec{v})_{<\alpha(\vec{v})} := j(z \mapsto B_i(\vec{v}^\frown \langle z \rangle))(u \restriction \alpha(\vec{v}))$ belongs to $\mathcal{F}(u \restriction \alpha(\vec{v}))$.

Similarly, we define the $u(\alpha(\vec{v}))$-large set

$$B_i(\vec{v})_{=\alpha(\vec{v})} := \{z \in \mathrm{Succ}_T(\vec{v}) \mid B_i(\vec{v}^\frown \langle z \rangle) = B_i(\vec{v})_{<\alpha(\vec{v})} \cap V_{\kappa_z}\}.$$

Finally, let $B_i(\vec{v})_{>\alpha(\vec{v})} := \{z \in A^{p^\omega} \mid \ell(z) > \alpha(\vec{v})\}$ and

$$B_i(\vec{v}) := B_i(\vec{v})_{<\alpha(\vec{v})} \cup B_i(\vec{v})_{=\alpha(\vec{v})} \cup B_i(\vec{v})_{>\alpha(\vec{v})}.$$

To amalgamate all of these $B_i(\vec{v})$ we take diagonal intersections; namely,

$$B_i := \{z \in \mathcal{MS} \mid \forall \vec{v} \in T \restriction i \, (\vec{v} \prec z \Rightarrow z \in B_i(\vec{v}))\}.$$

It is routine to check that $B_i \in \mathcal{F}(u)$.

In the end, we let $B := \bigcap_{i \leq \ell} B_i$ and $S := (\bigcap_{i \leq \ell} S_i) \cap B$. We claim that $p^* := \langle (u, B) \rangle$ together with S satisfy the statement of the claim. For this it suffices to show that if $\vec{w} \in \mathcal{B}(S)$ then $q_{\vec{w}} \leq^* p^{*\curvearrowright} \vec{w}$.

For each $\vec{w} \in \mathcal{B}(S)$ we have

$$p^{*\curvearrowright} \vec{w} := \langle (w_0, B_0^*), \ldots, (w_n, B_\ell^*), (u, B_{\ell+1}^*) \rangle$$

where $B_i^* := \{v \in B \cap V_{\kappa_{w_i}} \mid w_{i-1} \prec v \wedge \ell(v) < \ell(w_i)\}$ for $i \leq \ell + 1$.[5]

Let us check that $B_i^* \subseteq B_i(\vec{w})$ for $i \leq \ell$ – the argument showing $B_{\ell+1}^* \subseteq B(\vec{w})$ is similar. First, $\langle w_{i+1}, \ldots, w_\ell \rangle \in (S_i)_{\vec{w} \restriction i+1}$ so

$$B_i(\vec{w}) = B_i(\vec{w} \restriction i+1).$$

Second, $w_i \in \mathrm{Succ}_{S_i}(\vec{w} \restriction i) = \mathrm{Succ}_T(\vec{w} \restriction i) \in u(\alpha(\vec{w} \restriction i))$. In particular, $\ell(w_i) = \alpha(\vec{w} \restriction i)$. Now let $v \in B_i^*$. By definition of diagonal intersection,

$$v \in B \cap V_{\kappa_{w_i}} \subseteq B_i(\vec{w} \restriction i) \cap V_{\kappa_{w_i}}.$$

Also, v has length $<\alpha(\vec{w} \restriction i)$ so it belongs to

$$B_i(\vec{w} \restriction i)_{<\alpha(\vec{w} \restriction i)} \cap V_{\kappa_{w_i}} = B_i(\vec{w} \restriction i+1) = B_i(\vec{w}).$$

[5]Here we agree that $w_{-1} = w_{\ell+1}$ are the empty sequence.

For the first of these equalities we used that $w_i \in B_i(\vec{w} \restriction i)_{=\alpha(\vec{w}\restriction i)}$.

Thereby we have showed that $B^*_i \subseteq B_i(\vec{w})$ as sought. $\square$

The above claim completes the verification of the lemma. $\square$

Let us now describe the main combinatorial object introduced by $\mathbb{R}_u$:

Definition 4.15. Let $G \subseteq \mathbb{R}_u$ be a V-generic filter. Denote

- $\mathcal{MS}_G := \{v \in \mathcal{MS} \mid \exists p \in G\ \exists i < \ell(p)\ \ v = u^p_i\}$;
- $\Sigma_G := \{\sigma_v \mid v \in \mathcal{MS}_G\}$;
- $C_G := \{\kappa_v \mid v \in \mathcal{MS}_G\}$.

Proposition 4.16. *There is $\xi < \kappa$ such that $\Sigma_G \setminus \xi$ is a totally discontinuous sequence; namely,* $\sup(\Sigma_G \cap \alpha) < \alpha$ *for all $\alpha \in \Sigma_G \setminus \xi$.*

Moreover, for each such α, $\Sigma_G \cap \alpha \subseteq \pi(\alpha) < \alpha$.[6]

Proof. Let us go for the moreover assertion. Recall that $j\colon V \to M$ is a constructing embedding for u and that $j(\pi)(\sigma) = \kappa < \sigma$. It follows that $X = \{v \in \mathcal{MS} \mid \pi(\sigma_v) < \sigma_v\}$ belongs to $\mathcal{F}(u)$. In particular, the set of conditions p with $A^p_{\ell(p)} \subseteq X$ is $\leq^*$-dense. Let $p \in G$ be a condition with that property and define $\xi := \sigma_{u^p_{\ell(p)-1}}$. For each $\alpha \in \Sigma_G \setminus (\xi + 1)$ there is $v \in \mathcal{MS}_G$ such that $\alpha = \sigma_v$ (note that v must come from X). Let $q \in G$ witnessing this. For each $\beta \in (\Sigma_G \cap \alpha) \setminus (\xi + 1)$ we can let $q \leq r$ in G such that $\sigma_w = \beta$. Notice that because $\beta < \alpha$ it must be that w is mentioned in r before v (and, once again, $w \in X$). By definition of the poset this implies that $w \prec v$, which in turn yields $\beta = \sigma_w < \pi(\sigma_v) = \pi(\alpha)$, as needed. $\square$

Remark 4.17. On the contrary, standard arguments show that C_G is a club on κ. This is the Radin club introduced by the normal Radin induced by π.

Arguing similarly one can prove the next propositions:

Proposition 4.18. *Suppose that $\alpha < \ell(u) \leq \kappa$ and let $A \in \mathcal{F}(u)$. Then there is $\xi < \kappa$ such that $(\mathcal{MS}_G \cap \{v \in \mathcal{MS} \mid \ell(v) \geq \alpha\}) \setminus V_\xi \subseteq A$.*

Proposition 4.19. *Suppose that $v \in \mathcal{MS}_G$ is such that κ_v has limit index in the natural enumeration of C_G then $\ell(v) > 1$.*

Corollary 4.20. *Suppose that $2 < \ell(u) \leq \kappa$ and let $A \in \mathcal{F}(u)$. Then there is $\xi < \kappa$ such that $\{\kappa_v \in C_G \mid v$ has limit index in $C_G\} \setminus \xi \subseteq \{\kappa_v \mid v \in A\}$.*

5. Adding Cohen functions to every limit point

Suppose that κ is a $\mathcal{P}_2\kappa$-hypermeasurable cardinal. In this section we employ our Radin forcing from §4 to shoot a club $C \subseteq \kappa$ with $\mathrm{otp}(C) = \omega_1$ whose limit points α carry a Cohen generic set $c_\alpha \subseteq \alpha$. Our main result here is Theorem 5.4. The next prelimminary result paves the way to Theorem 5.4.

Lemma 5.1. *Assume the* GCH *holds and that κ is $\mathcal{P}_2\kappa$-hypermeasurable cardinal. There is a cofinality-preserving generic extension $V[G]$ where:*

[6] Recall that $\pi\colon \kappa \to \kappa$ is the function representing κ_u via σ_u (see p.14).

(1) GCH *holds;*
(2) There is a $\mathcal{P}_2\kappa$*-hypermeasurable embedding* $j : V[G] \to M[H]$ *and an ordinal* $\sigma \in (\kappa, j(\kappa))$ *such that the (non-normal) measure*

$$W := \{X \in \mathcal{P}(\kappa)^{V[G]} \mid \sigma \in j(X)\}$$

witnesses that its Tree Prikry forcing $\mathbb{T}_W$ *projects onto* $\mathrm{Add}(\kappa, 1)$.

Proof. Let us begin by fixing an elementary embedding $j\colon V \to M$ arising from a (κ, κ^{++})-extender E – this is possible in that κ is $\mathcal{P}_2\kappa$-hypermeasurable. Let us denote by $i\colon V \to N$ the ultrapower by the normal measure on κ inferred from j. As usual, this yields a factor embedding $k\colon N \to M$ defined by $k(i(f)(\kappa)) := j(f)(\kappa)$. Standard arguments involving the GCH show that k has width $\leq\kappa_N^{++}$ and that $\mathrm{crit}(k) = \kappa_N^{++}$.

We go for the forcing preparation spelled out in [BG21, § 7]. Namely, our forcing extension will be given by the Easton-supported iteration

$$\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \kappa,\ \beta < \kappa\rangle$$

defined as follows: For each $\alpha < \kappa$, $\dot{\mathbb{Q}}_\alpha$ is a $\mathbb{P}_\alpha$-name for the trivial forcing unless α is inaccessible, in which case it is a $\mathbb{P}_\alpha$-name for the lottery sum of

$$\{\mathrm{Add}(\alpha, 1), \{\mathbb{1}\}\}.$$

Let $G \subseteq \mathbb{P}_\kappa$ a V-generic filter. It is easy to lift the embeddings $i\colon V \to N$ and $k\colon N \to M$ inside $V[G]$. Indeed, i lifts to $i\colon V[G] \to N[G * \{\mathbb{1}\} * H]$ where $H \in V[G]$ is a $N[G]$-generic filter for the tail poset $i(\mathbb{P}_\kappa)/G * \{\mathbb{1}\}$.[7] Since this tail forcing is more close than the width of the embedding k one can lift this latter to $k\colon N[G * \{\mathbb{1}\} * H] \to M[G * \{0\} * k``H]$. Incidentally,

$$j\colon V[G] \to M[G * \{0\} * k``H].$$

For reasons that will become clear shortly we have to prepare a $M[j(G)]$-generic filter $g_{j(\kappa)}$ for $\mathrm{Add}(j(\kappa), 1)_{M[j(G)]}$. This is done as before: First, one can cook up a $N[i(G)]$-generic $g_{i(\kappa)} \in V[G]$ for $\mathrm{Add}(i(\kappa), 1)_{N[j(G)]}$ (for this one employs the GCH). Second, we can trasnfer $g_{i(\kappa)}$ to $g_{j(\kappa)}$ through k; clearly, $g_{j(\kappa)} \in V[G]$. In addition, we can alter $g_{j(\kappa)}$ so that $g_{j(\kappa)}(0) = \kappa$.

Now we go to the second ultrapower of our initial extender E; specifically, let us consider $j_{1,2}\colon M \to M_2 \simeq \mathrm{Ult}(M, j(E))$. Eventually, we would like to lift $j_2 := j_{1,2} \circ j$ and for this it would suffice to lift $j_{1,2}$ under $j(\mathbb{P}_\kappa)$ (as the other embedding has been already lifted). To this end, note that

$$j_2(\mathbb{P}_\kappa) \downarrow p \simeq j(\mathbb{P}_\kappa) * \dot{\mathrm{Add}}(j(\kappa), 1) * \dot{\mathbb{T}}_{(j(\kappa), j_2(\kappa))}$$

where p is the condition opting for Cohen forcing at stage $j(\kappa)$.

Clearly, $\mathrm{Add}(j(\kappa), 1)_{M_2[j(G)]} = \mathrm{Add}(j(\kappa), 1)_{M_1[j(G)]}$ so $j_{1,2}$ lifts to

$$j_{1,2}\colon M[j(G)] \to M_2[j(G) * g_{j(\kappa)} * T]$$

[7]The construction of a H in $V[G]$ is standard employing the GCH and the high degree of closure of the tail forcing.

for some generic T for the tail forcing. As before, we can construct this T inside $M[j(G)]$ by factoring through the normal ultrapower of $j_{1,2}$ and using the GCH in the model $M[j(G)]$. Therefore, the above lives inside $V[G]$.

This produces an elementary embedding $j_2\colon V[G] \to M_2[j_2(G)]$ such that:

Claim 5.2. *The following hold for j_2 in $V[G]$:*

(1) j_2 is a $\mathcal{P}_2\kappa$-hypermeasurable embedding;
(2) Let $W := \{X \in \mathcal{P}(\kappa)^{V[G]} \mid j(\kappa) \in j_2(X)\}$. Then, $\mathrm{Cub}_\kappa \subseteq W$ and

$$\mathcal{C} := \{\alpha < \kappa \mid \exists f_\alpha\, (f_\alpha \text{ is Cohen generic over } V[G_\alpha])\} \in W;$$

(3) $j_2(\pi)(j(\kappa)) = \kappa$ where $\pi\colon \kappa \to \kappa$ is the function defined as

$$\pi(\alpha) := \begin{cases} f_\alpha(0), & \text{if } \alpha \in \mathcal{C}; \\ 0, & \text{otherwise.} \end{cases}$$

Proof of claim. (1) Note that $V_{\kappa+2} \subseteq M_2$ because $V_{\kappa+2} \subseteq M_1$ and both M_1 and M_2 agree up to $(V_{j(\kappa)})^{M_1}$. Also, $\mathbb{P}_\kappa$ is κ-cc so that $V[G]_{\kappa+2} = V_{\kappa+2}[G]$ From this we can easily infer that $V[G]_{\kappa+2} \subseteq M_2[j_2(G)]$.

(2) Let $C \in (\mathrm{Cub}_\kappa)^{V[G]}$. By κ-ccness of $\mathbb{P}_\kappa$ there is $D \subseteq C$ in $(\mathrm{Cub}_\kappa)^V$. By normality, $j(D)$ belongs to the normal measure on $j(\kappa)$ inferred (in V) from $j_{1,2}$. From this it follows right away that $C \in W$.

The claim about $\mathcal{C}$ is evident because $g_{j(\kappa)}$ was chosen to be a Cohen generic over $M_2[j(G)] = M_2[j_2(G)_{j(\kappa)}]$.

(3) This follows from our choice that $g_{j(\kappa)}(0) = \kappa$. □

To complete proof of Lemma 5.1 it remains to show that (in $V[G]$) the Tree Prikry forcing $\mathbb{T}_W$ corresponding to W projects onto $\mathrm{Add}(\kappa, 1)_{V[G]}$.

Claim 5.3. *There is a projection between $\mathbb{T}_W$ and $\mathrm{Add}(\kappa, 1)_{V[G]}$.*

Proof. Let $\langle \kappa_n \mid n < \omega\rangle$ be a Prikry sequence over $V[G]$. We shall show that this induces a $V[G]$-generic for the Cohen poset. Let $A \in V[G]$ be a maximal antichain for $\mathrm{Add}(\kappa, 1)_{V[G]}$. For each $\alpha < \kappa$ regular and $p \in \mathrm{Add}(\alpha, 1)_{V[G]}$ let $\beta(p) < \alpha$ be the first ordinal such that p is compatible with a member of $A \cap \mathrm{Add}(\beta(p), 1)_{V[G]}$. Define a function $f\colon \kappa \to \kappa$ by

$$f(\alpha) := \sup\nolimits_{p \in \mathrm{Add}(\alpha,1)_{V[G]}} \beta(p)$$

whenever α is a regular cardinal; declare it to be 0 otherwise.

Let $C(f)$ be the closure points of f. This set is a club in $V[G]$ so $C \cap \mathcal{C} \in W$ (here $\mathcal{C}$ is as in the previous claim). Let $1 \leq n_* < \omega$ be such that $\kappa_n \in C \cap \mathcal{C}$ for all $n \geq n_*$ and define (in $V[G]$)

$$f^* := \bigcup\nolimits_{n \geq n_*} f_{\delta_n} \restriction [\delta_{n-1}, \delta_n).$$

One can show that f^* is $\mathrm{Add}(\kappa, 1)_{V[G]}$-generic. We refer the reader to [BG21, Proposition 7.3] for details. □

The above claim completes the proof of the lemma. □

Let V^* denote the model obtained in Lemma 5.1 and $j^*\colon V^* \to M^*$ the corresponding $\mathcal{P}_2\kappa$-hypermeasurable embedding. From now on V^* will be our ground model. Using j^*, π and σ from Lemma 5.1 we derive the corresponding measure sequence u; namely, $u(0) := \langle\sigma\rangle$ and for each $\xi \geq 1$,

$$u(\xi) := \{X \subseteq V_\kappa \mid u \upharpoonright \xi \in j^*(X)\}.$$

Note that $u(1)$ is essentially W; more precisely,

$$X \in W \text{ if and only if } \{\langle\alpha\rangle \mid \alpha \in X\} \in u(1).$$

Theorem 5.4. *Let $G^* \subseteq \mathbb{R}_u$ be V^*-generic. For all except bounded-many $\alpha \in \lim(C_{G^*}) \cup \{\kappa\}$ with $\mathrm{cf}(\alpha)^{V^*[G^*]} = \omega$ there is a V^*-generic Cohen function $f_\alpha \in V^*[G^*]$ for $\mathrm{Add}(\alpha,1)_{V^*}$.*

Proof. Let us begin noting that[8]

$$X := \{v \in \mathcal{MS} \mid \exists\pi\colon \mathbb{T}_{v(1)} \to \mathrm{Add}(\kappa_v,1)\,\text{projection in } V^*\} \in \mathcal{F}(u).$$

Indeed, this is because $\mathbb{P}(W)$ projects onto $\mathrm{Add}(\kappa,1)$ and this is correctly computed by the model M^*. By Proposition 4.18 there is $\beta < \kappa$ such that

$$(\mathcal{MS}_{G^*} \cap \{v \in \mathcal{MS} \mid \ell(v) > 1\}) \setminus V_\beta \subseteq X.$$

Let $\alpha \in \lim(C_{G^*}) \cup \{\kappa\}$ be with $\alpha > \beta$ and $\mathrm{cf}(\alpha)^{V[G^*]} = \omega$. By definition there is $v \in \mathcal{MS}_G$ such that $\alpha = \kappa_v$ and, clearly, α must have limit index in the enumeration of C_{G^*}. By Proposition 4.19, v is a measure sequence with $\ell(v) > 1$ so the above inclusion gives $v \in X$. Thus, $\mathbb{T}_{v(1)}$ projects onto $\mathrm{Add}(\kappa_v,1)$. Next we show that a bounded piece of the Radin club $\langle\kappa_\alpha \mid \alpha < \omega^{\ell(u)}\rangle$ can be used to produce a generic for $\mathrm{Add}(\kappa_v,1)$.

Let $\langle v_n \mid n < \omega\rangle \subseteq \mathcal{MS}_{G^*}$ of length 1 such that $\sup_{n<\omega}\kappa_{v_n} = \alpha$.

Claim 5.5. *$\langle\alpha_n \mid n < \omega\rangle$ is a $\mathbb{T}_{v(1)}$-generic sequence.*

Proof of claim. Let us use the Mathias criterion for the Tree-Prikry forcing from [Ben19]. Let $A \in v(1)$ and $p \in G^*$ be condition mentioning v; say at coordinate i. Shrink A^p_i to $A^*_i \subseteq A$, and extend $p \leq^* p^*$ so that the i-th set in p^* is A^*_i. Note that p^* forces that every successor element element of C_{G^*} in the interval $(\sigma_{\bar{u}^p_{i-1}}, \alpha)$ is in A. By density, we can find such a condition in G^* and so a tail of the α_n's falls in A. □

Since $\mathbb{T}_{v(1)}$ projects on $\mathrm{Add}(\alpha,1)$, there is a generic Cohen function $f \in V^*[\langle\alpha_n \mid n < \omega\rangle]$ and since $V^*[\langle\alpha_n \mid n < \omega\rangle] \subseteq V^*[G^*]$ we are done. □

Corollary 5.6. *If $\ell(u) = \omega_1$ then below a certain condition $p \in \mathbb{R}_u$ the poset $\mathbb{R}_u/p$ adds a V^*-generic Cohen function to every limit point of the generic club C_G.*

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[8] In a slight abuse of notation, here we have identified $v(1)$ with the corresponding measure on κ_v rather than on V_{κ_v}.

After an appropriate preparation, we have just shown that forcing a $\mathbb{R}_u$-generic club $C \subseteq \kappa$ of order-type ω_1 automatically adds a V-generic Cohen subset to every limit point of C. Clearly, all those points have countable cofinality in $V[C]$. However, do we add a Cohen subset to κ? Or, alternatively, if we employ $\mathbb{R}_u$ to add a generic club $C \subseteq \kappa$ of order-type ω_2, do the limit points of C of cofinality ω_1 carry a V-generic Cohen subset in $V[C]$? Suppose that $\vec{\kappa} = \langle \kappa_\alpha \mid \alpha < \omega_1 \rangle$ is a Magidor/Radin generic. If one attempts to amalgamate a Cohen generic f using $\vec{\kappa}$ –similarly to what we did in Claim 5.3– this will not work: On one hand, the restriction $f \restriction \kappa_\omega$ is generic over the ground model by virtue of Claim 5.3. On the other hand, if f is a V-generic for $\mathrm{Add}(\kappa, 1)$ then $f \restriction \alpha \in V$ for all $\alpha < \kappa$. This restriction is in fact hiding the impossibility for a Radin-like forcing to introduce a fresh subset of κ, provided this latter cardinal changes its cofinality to $\geq\omega_1$.

Definition 5.7. Let $\mathbb{P}$ be a forcing poset and $G \subseteq \mathbb{P}$ a V-generic filter. A set $x \subseteq \kappa$ is called $(V, V[G])$*-fresh* if $x \in V[G]$ and $x \cap \alpha \in V$ for all $\alpha < \kappa$.

The next fact appears in [BN19, Proposition 1.3] where the author gives credit to Cummings and Woodin. The proof in the non-normal scenario is verbatim the same as the one provided by Ben-Neria – one simply replaces the usual diagonal intersection by our revised definition employing the order $\prec$ of Definition 4.4:

Fact 5.8 (Cummings and Woodin)**.** *Assume $u \in \mathcal{MS}$ has $\mathrm{cf}(\ell(u)) \geq \omega_1$. Then, the trivial condition of $\mathbb{R}_u$ forces that "$\forall \tau \subseteq \kappa\, (\tau$ is fresh $\Rightarrow \tau \in \check{V})$".*

The same holds true for the non-normal Magidor forcing defined in §3.

Corollary 5.9. *If x is $(V, V[G])$-fresh then $\mathrm{cf}^{V[G]}(\sup(x)) = \omega$.*

Proof. The proof is by induction on $\sup(x)$. Denote by $\lambda = \mathrm{cf}^V(\sup(x))$ and let $\langle \delta_\alpha \mid \alpha < \lambda \rangle \in V$ be a cofinal sequence in $\sup(x)$.

<u>Case $\lambda \geq \kappa^+$</u>: For each $\alpha < \lambda$, since x is fresh, we can let $p_\alpha = \vec{d_\alpha}{}^\frown(u, A_\alpha) \in G$ deciding the value of $\dot{x} \cap \delta_\alpha$. By passing to an unbounded subset of λ we can assume that $\vec{d_\alpha} = \vec{d_*}$. Next define

$$y = \{\nu < \kappa \mid \exists A \in \mathcal{F}(u),\ \vec{d_*}{}^\frown(u, A) \Vdash \nu \in \dot{x}\}.$$

Then $y \in V$ and we claim that $y = x$. Indeed, if $\nu \in y$ then, for some $\alpha < \lambda$, $\nu < \delta_\alpha$ and there is A such that $p' = \vec{d_*}{}^\frown(u, A) \Vdash \nu \in \dot{x}$. Since $\vec{d_*}{}^\frown(u, A_\alpha) \Vdash \dot{x} \cap \delta_\alpha = x \cap \delta_\alpha$ is compatible with p', it must be that $\nu \in x \cap \delta_\alpha$ (otherwise, a common extension would have forced contradictory information). Conversely, if $\nu \in x$ one finds δ_α such that $\nu < \delta_\alpha$. Since $\vec{d_*}{}^\frown(u, A_\alpha) \Vdash \nu \in x \cap \delta_\alpha = \dot{x} \cap \delta_\alpha$, it follows that A_α witness that $\nu \in y$.

<u>Case $\lambda \leq \kappa$</u> Let $x = x_0$. We fix in V a sequence $\langle \phi_\alpha \mid \alpha < \lambda \rangle \in V$ such that $\phi_\alpha : \mathcal{P}(\delta_\alpha) \to 2^{\delta_\alpha}$ is a bijection. Let $\lambda_\alpha = \phi_\alpha(x \cap \delta_\alpha)$. By κ^+-c.c of $\mathbb{R}_u$, we can find $f : \lambda^* \to \mathcal{P}_{\kappa^+}(\lambda^*) \in V$ (where $\lambda^* = \sup\{\lambda_\alpha \mid \alpha < \lambda\}$) such that $\lambda_\alpha \in f(\alpha)$. For each α let $i_\alpha < \kappa$ be such that λ_α is

the i_α-th element of $f(\alpha)$ in its increasing enumeration. We can define i^*_α recursively as follows $i^*_0 = i_0$ and $i^*_\alpha = (\sup_{j<\alpha} i^*_j) + i_\alpha$. Note that i^*_α is increasing and i_α is definable from the sequence i^*_α (as the unique ordinal γ such that $(\sup_{j<\alpha} i^*_j) + \gamma = i^*_\alpha$). Also note that since κ is regular in V, and for each $\beta < \lambda \leq \kappa$, $\{i_\alpha \mid \alpha < \beta\} \in V$, $i^*_\alpha < \kappa$. We conclude that the set $x_1 = \{i^*_\alpha \mid \alpha < \lambda\} \subseteq \kappa$ is fresh. If x_1 is unbounded in κ, then by the previous proposition, $\omega = \mathrm{cf}^{V[G]}(\kappa) = \mathrm{cf}^{V[G]}(\lambda) = \mathrm{cf}^{V[G]}(\sup(x))$. Otherwise, x_1 is bounded in κ and we let $\kappa^*_1 = \sup(\lim(C_G) \cap \sup(x_1)) < \kappa$. Then $x_1 \in C_G \restriction \kappa^*_1$, and we may apply the induction hypothesis. $\square$

For a forcing notion $\mathbb{Q}$ let us denote by $\mathrm{dist}(\mathbb{Q})$ the unique λ such that $\mathbb{Q}$ is λ-distributive yet not λ^+-distributive. Equivalently,

$$\mathrm{dist}(\mathbb{Q}) = \min\{\theta \in \mathrm{Card} \mid \exists \tau \in V^{\mathbb{Q}}\ \mathbb{1} \Vdash_{\mathbb{Q}} \text{“}\tau \subseteq \mathrm{Ord} \wedge |\tau| = \theta \wedge \tau \notin \check{V}\text{”}\}.$$

Note that $\mathrm{dist}(\mathbb{Q})$ is a regular cardinal in $V^{\mathbb{Q}}$.

Corollary 5.10. $\mathbb{R}_u$ *projects only on forcings* $\mathbb{Q}$ *such that* $\mathrm{cf}^{V[G]}(\mathrm{dist}(\mathbb{Q})) = \omega$ *and therefore* $\mathrm{dist}(\mathbb{Q}) \in \{\omega\} \cup (\lim(C_G) \cap \mathrm{cf}(\omega))$.

Proof. Suppose $\mathrm{dist}(\mathbb{Q}) = \lambda$.

Claim 5.11. *There is a fresh set of ordinals* $A \in V^{\mathbb{Q}} \setminus V$ *such that*

$$\mathrm{cf}^{V^{\mathbb{Q}}}(\sup(A)) = \lambda.$$

Proof of claim. Let $A \in V^{\mathbb{Q}} \setminus V$ be a set of ordinals with $\lambda = |A|^{V^{\mathbb{Q}}}$. Take $\rho \leq \sup(A)$ be the minimal ordinal such that $A \cap \rho \notin V$. If $\mathrm{cf}^{V^{\mathbb{Q}}}(\rho) < \lambda$ we would reach a contradiction with the fact of $\mathbb{Q}$ being λ distributive. Hence it must be that $\mathrm{cf}^{V^{\mathbb{Q}}}(\rho) \geq \lambda$, in which case, $\mathrm{cf}^{V^{\mathbb{Q}}}(\rho) = \lambda$ since $A \cap \rho \in V^{\mathbb{Q}}$ is of size $\leq \lambda$ and must be unbounded in ρ (by minimality of ρ). $\square$

Let A be a set as in the claim. By Corollary 5.9, $\mathrm{cf}^{V[G]}(\sup(A)) = \omega$, and as a result $\mathrm{cf}^{V[G]}(\lambda) = \omega$. Hence λ is a regular cardinal which changed its cofinality in $V[G]$ to ω, and thus $\lambda \in \{\omega\} \cup (\lim(C_G) \cap \mathrm{cf}(\omega))$. $\square$

Recall that by Theorem 3.19 the non-normal Magidor forcing of §3 is a projection of the extender based Magidor-Radin forcing. Similarly, it is possible to show that the non-normal Radin forcing of this section is a projection of the extender-based Radin forcing from [Mer03a].

Let us use the observation above regarding our forcing to conclude that also in the extender-based Radin and Magior/Radin there are no fresh subsets of κ. We will need to use the properness-like property of the extender-base Magidor radin forcing [Mer11, Lemma 4.13]: Assume χ is large enough, $N \prec H_\chi$ is an elementary submodel, and $P \in N$ is a forcing notion. A condition $p \in P$ is called $\langle N, P\rangle$-generic if for each dense open subset $D \in N$ of P,

$$p \Vdash_P \check{D} \cap \underset{\sim}{G} \cap \check{N} \neq \emptyset$$

where $\underset{\sim}{G}$ is the name of the P-generic object.

Corollary 5.12. *Suppose that $\bar{E} = \langle E_\xi \mid \xi < o(\bar{E})\rangle$ is an extender sequence with $\mathrm{cf}(o(\bar{E})) \geq \omega_1$ such that each E_ξ is a (κ, λ_ξ)-extender and $\lambda_\xi < j_{E_0}(\kappa)$. Let $\mathbb{P}_{\bar{E}}$ be either the extender-based Radin forcing or the extender-based Magidor/Radin forcing. Then, for every V-generic filter $G \subseteq \mathbb{P}_{\bar{E}}$ there are no $(V, V[G])$-fresh subsets of κ.*

Proof. Let us prove that if $A \subseteq \kappa$, $A \in V[G]$, then there is a sequence of α_i's such that $\alpha_i < j_{E_0}(\kappa)$, and $\langle E_i(\alpha_i) \mid i < o(\bar{E})\rangle$ is $\triangleleft$-increasing, such that $A \in V[G^*]$, where G^* is the projected generic for $\mathbb{M}[\vec{U}]$, $\vec{U}$ being the generalized cohere sequence derived from $\langle E_i(\alpha_i) \mid i < o(\bar{E})\rangle$. Since $V[G^*]$ does not have fresh subsets of κ, A cannot be $(V, V[G])$-fresh. Let $\langle \underset{\sim}{a}_i \mid i < \kappa\rangle$ be a sequence of $\mathbb{P}_{\bar{E}}$-names for an enumeration of A and let $N \prec H_\chi$ $|N| = \kappa$, $\langle \underset{\sim}{a}_i \mid i < \kappa\rangle, \mathbb{P}_{\bar{E}} \in N$, N is closed under $< \kappa$-sequences and $N \cap \kappa^+ \in \kappa^+$. Then there is p^* which is $(N, \mathbb{P}_{\bar{E}})$-generic [Mer11, Lemma 4.13]. In particular, consider the dense open set

$$D_i = \{p \in \mathbb{P}_{\bar{E}} \mid p \text{ decides } \underset{\sim}{a}_i\}$$

then $D_i \in N$ since $\underset{\sim}{a}_i, \mathbb{P}_{\bar{E}} \in N$ and by elementarity. Let $Y = N \cap \mathfrak{D}$, then $Y \in P_{\kappa^+}(\mathfrak{D})$. Let us find in G an $(N, \mathbb{P}_{\vec{E}})$-generic condition $p^* \in G$.

Claim 5.13. *For each $i < o(\bar{E})$, it is possible to find a single $\alpha_i < j_{E_0}(\kappa)$ and a function f_i, such that $j_{E_i}(f_i)(\alpha_i) = mc_i(Y)$ and $\langle E_i(\alpha_i) \mid i < o(\bar{E})\rangle$ is $\triangleleft$-increasing.*

Proof. Fix $i < o(\bar{E})$. Find a bijection $\phi : \kappa \to [\kappa]^{<\omega}$ such that for every limit ordinal α of cofinality $|\alpha|$ $\phi \restriction \alpha : \alpha \to [\alpha]^{<\alpha}$. We construct α_i's by induction. In M_{E_i}, represent $j_{E_i}(f_i)(\xi_1, ...\xi_n) = Y$, where $\xi_1, ..., \xi_n < \lambda_i < j_{E_0}(\kappa)$ $j_{E_i}(g_i)(\eta_1, ...\eta_m) = \{\alpha_j \mid j < i\}$ and $j_{E_i}(h_i)(\zeta_1, ..., \zeta_k) = \langle E_j \mid j < i\rangle$. Let $\alpha_i = j_{E_i}(\phi)(\{\xi_1, ..., \xi_m, \eta_1, ..., \eta_m, \zeta_1, ..., \zeta_k\})$. Using the fact that $o(\bar{E})$ is small, we see that $E_j(\alpha_j) \in M_{E_i(\alpha_i)}$ for all $j < i$ and that there is a function f'_i such that $j_{E_i}(f'_i)(\alpha_i) = mc_i(Y)$. □

We have $R_i \in E_i(Y \cup \{\bar{\alpha}_i\})$, such that for each $\mu \in R_i$, $\bar{\alpha}_i \in \mathrm{dom}(\mu)$, $o(\mu) = i$, and $\mu \restriction (Y \cap \mathrm{dom}(\mu)) = f_i(\mu(\bar{\alpha}_i)_0)$. So we can find $p^* \leq^* p_* \in G$. To simplify the notation let us assume that $p_* = \langle \bigcup_{i<o(\bar{E})} R_i, f_0, \bar{E}\rangle$. Let us define $q \restriction Y$ and $Y_i^q = \mathrm{dom}(f_i^{q \restriction Y}) \subseteq \mathrm{dom}(f_i^q)$ for each $1 \leq i \leq l(q) + 1$ for every $q \leq p_*$. The indented meaning of Y_j^q is the collection of extender sequences indexing the *jth-block* of a condition q which extends a pure condition whose top block is indexed by Y. Recall that when we extend a condition we have to reflect/squeeze the extender sequences indexing each block – this is exactly the meaning of Y_j^q.

$p_* \restriction Y = p_*$ and $Y_1^{p^*} = Y$. Suppose that $q \restriction Y$ and Y_i^q's were defined, let $\mu \in A_i^q$, define $(q^\frown\mu) \restriction Y = q \restriction Y^\frown(\mu \restriction Y_i^q)$ and

$$Y_j^{q^\frown\mu} = \begin{cases} Y_j^q & j < i. \\ \mu[Y_i^q] & j = i. \\ Y_{j-1}^q & j > i. \end{cases}.$$

Since $Y_i^q \subseteq \mathrm{dom}(f_i^{q\restriction Y})$, $\mu \restriction Y_i^q \in A_i^{q\restriction Y}$. If $q' \leq^* q$ we define $A_i^{q'\restriction Y} = \{\mu \restriction Y_i^q \mid \mu \in A_i^{q'}\}$ and $Y_j^q = Y_j^{q'}$. Note that for every q, $q \restriction Y$ is a condition and that the map $q \mapsto q \restriction Y$ respects both $\leq$ and $\leq^*$.

Claim 5.14. *If $r \in N$ and $r \leq q$ then $r \leq q \restriction Y$.*

Proof. By induction on $l(q)$. For $l(q) = 1$, and $r \in N$ such that $r \leq q$ we have $r = \langle\langle f^r, A^r, \bar{E}\rangle\rangle$. Since $r \in N$, $f^r \in N$ and $N \cap \kappa^+ \in \kappa^+$, $\mathrm{dom}(f^r) \subseteq N \cap \mathfrak{D} = Y$. This suffices to infer that $r \leq q \restriction Y$.

Let us provide details for the case $l(q) = 2$ (the others are analogue by induction). In that case

$$q = q_0{}^\frown\mu = \langle\langle f_1^q, A_1^q, e_1^q\rangle, \langle f_2^q, A_2^q, \bar{E}\rangle\rangle.$$

If $r \leq q$, $r \in N$, we may assume that $l(r) = 2$ for otherwise we can apply the induction hypothesis. So

$$r = \langle\langle f_1^r, A_1^r, e_1^r\rangle, \langle f_2^r, A_2^r, \bar{E}\rangle\rangle = r_0{}^\frown\mu'$$

where $\mu' = \mu \restriction \mathrm{dom}(f_2^r)$. Thus

$$\mathrm{dom}(f_1^r) = \{\mu'(\bar{\alpha}) \mid \bar{\alpha} \in \mathrm{dom}(\mu'),\ o(\bar{\alpha}) > 0\} \subseteq \mu'[Y] \subseteq \mu[Y] = Y_1^q.$$

So $r \leq q \restriction Y$. □

For each $i < \kappa$, let $p_i \in G \cap D_i \cap N$, then there is $p_i^* \in G$ which is a common extension of p_* and p_i, and we consider $q_i = p_i^* \restriction Y$. By the claim $p_i \leq q_i$ and therefore $q_i \in D_i \cap G \restriction Y$ where $G \restriction Y = \{q \restriction Y \mid q \in G/p_*\}$.

Let G^* be the V-generic induced from G for $\mathbb{M}[\vec{U}]$, and $\vec{U}$ is the generalized coherent sequence induced from

$$\langle E_i(\bar{\alpha}_i) \mid i < o(\bar{E})\rangle.$$

We shall now prove that $G \restriction Y \in V[G^*]$ and then we can choose in $V[G^*]$, $p_i' \in G \restriction Y \cap D_i$ which suffices to compute $A \in V[G^*]$, as $G \restriction Y \subseteq G \cap D_i$.

For any condition $p \in \mathbb{M}[\vec{U}]$, we define $p' \in \mathbb{P} \restriction Y$ such that $l(p) = l(p')$ along with functions $g_{i,j}^p$ recursively as follows:

If $p = \langle\langle \kappa, A\rangle\rangle$ we define $p' = \langle\langle f, B, \bar{E}\rangle\rangle$ where $\mathrm{dom}(f) = Y$ $f(\bar{\alpha}) = \emptyset$ for every $\bar{\alpha}$ and $B = \bigcup_{i<o(\bar{E})}\{\mu \restriction Y \cap \mathrm{dom}(\mu) \mid \mu \in R_i, f_i(\mu(\bar{\alpha}_i)_0) \in A\}$. Also $g_{i,j}^p = f_j$. Suppose that p' and $g_{i,j}^p$ where defined and consider $p^\frown\beta$ where $\beta \in A_j^p$ with $o^{\vec{U}}(\beta) = i$. Then let $\mu_\beta = g_{i,j}^p(\beta)$ and let $(p^\frown\beta)' = p'{}^\frown\mu_\beta$ and $g_{i,j}^{p^\frown\beta} = g_{i,j}^p \circ \mu_\beta^{-1}$ (for the relevant j). For direct extensions, we just shrink the measure one set. Since $G \restriction Y$ is above p_*, the definition of the R_i ensures we can recover all of $G \restriction Y$ from G^*. For more details see the argument of [BG23, Thm. 4.2]. □

6. Gitik's forcing project onto Cohen forcing

In the previous section we demonstrated that the natural generalizations of Magidor/Radin forcing to the non-normal context do not introduce fresh subsets to a measurable cardinal κ provided this latter changes its cofinality to ω_1 in the corresponding generic extension. As a result none of these posets project onto any κ-distributive – including among them Cohen forcing $\mathrm{Add}(\kappa,1)$. This raises an obvious question: Suppose that $\mathbb{P}$ is a cardinal-preserving forcing changing the cofinality of a measurable κ to ω_1. Is it feasible at all for $\mathbb{P}$ to project onto $\mathrm{Add}(\kappa,1)$? In this section we show that (once again, after a suitable preparation) the natural non-normal version of Gitik's forcing from [Git86] does project onto $\mathrm{Add}(\kappa,1)$. We begin with a warm-up section §6.1 showing how to add a Cohen function along an ω^2-sequence. Later, in §6.2 we handle the case of interest; namely, we show how to add a Cohen function along an ω_1-sequence.

6.1. **Adding a Cohen function along an ω^2-sequence.** Let us denote our ground model by V_0. For the rest of this section, we shall suppose that the GCH holds in V_0 and that this latter model accommodates a measurable cardinal κ with $o(\kappa)=2$. Fix $U_0 \lhd U_1$ normal measures over κ. We begin performing the preparation from [BG21] – similarly to what we already did in Lemma 5.1. Namely, we force with the Easton-supported iteration $\mathbb{P}_\kappa$ forcing with the Lottery sum of $\mathrm{Add}(\alpha,1)$ and $\{\mathbb{1}\}$ for inaccessibles $\alpha<\kappa$.

Suppose that $G\subseteq \mathbb{P}_\kappa$ is V_0-generic. Then we can lift $j_{U_1}\colon V_0\to M_{U_1}$ to

$$j^*_{U_1}\colon V_0[G]\to M_{U_1}[j^*_{U_1}(G)]\subseteq V_0[G]$$

by letting the lottery to force trivially at κ. Standard arguments show that this is the ultrapower embedding by a normal measure W_1 extending U_1.

Let us write $j^*_{U_1}(G)=G*G_{(\kappa,j_{U_1}(\kappa))}$. Arguing as in [BG21] we lift the measure U_0 (within $M_{U_1}[G]$) to a non-normal measure W_0 such that:

Setup 1.

1. $\mathrm{Cub}^{V_0[G]}_\kappa\subseteq W_0$;
2. Forcing with the Tree Prikry forcing $\mathbb{T}_{W_0}$ yields a map

   $$f^*_\kappa\colon \kappa\to\kappa$$

   such that if $\langle \kappa_n\mid n<\omega\rangle$ is a Tree-Prikry generic sequence then

   $$f^*_\kappa := \bigcup_{n<\omega} f_{\kappa_n}\restriction [\kappa_{n-1},\kappa_n)$$

   is $M_{U_1}[G]$-generic for $\mathrm{Add}(\kappa,1)^{V[G]}$.

Notice that $j^*_{U_1}(\mathbb{P}_\kappa)/G$ does not add subsets to κ and as a result W_0 remains a measure on $M_{U_1}[j^*_{U_1}(G)]$ which contains the club filter $\mathrm{Cub}^{V_0[G]}_\kappa$.

Note that f^*_κ remains generic over $M_{U_1}[j^*_{U_1}(G)]$ because the tail forcing does not add new dense open subsets to the forcing. Similarly, the same applies to $V_0[G]$ as $M_{U_1}[j^*_{U_1}(G)]$ and $V_0[G]$ agree on $(V_0[G])_{\kappa+1}$.

In summary, we have produced two measures $W_0 \lhd W_1$ such that W_1 is normal, W_0 is non-normal yet contains $\mathrm{Cub}_\kappa^{V_0[G]}$ (i.e., W_0 is a Q-point) and forcing with $\mathbb{T}_{W_0}$ over $V_0[G]$ introduces an $\mathrm{Add}(\kappa,1)^{V_0[G]}$-generic (i.e., f^*_κ).

Convention 6.1. Herefarter we denote by V the prepared model $V_0[G]$.

We follow Gitik's work [Git86, §3] closely. We need a further preparation over V. Let $\alpha \mapsto W_{0,\alpha}$ be a function representing W_0 in M_{W_1}; namely, $j_{U_1}(\alpha \mapsto W_{0,\alpha})(\kappa) = W_0$. Let $A \in W_1$ witnessing the following:

(1) $W_{0,\alpha}$ is a measure on α and if $b_\alpha := \langle \kappa^\alpha_n \mid n < \omega\rangle$ is generic for $\mathbb{T}_{W_{0,\alpha}}$

$\mathbb{1} \Vdash_{\mathbb{T}_{W_{0,\alpha}}}$ "$\dot{f}^*_\alpha = \bigcup_{n<\omega} f_{\dot\kappa^\alpha_n} \restriction [\dot\kappa^\alpha_{n-1}, \dot\kappa^\alpha_n)$ is V-generic for $\mathrm{Add}(\alpha,1)^V$".

(2) $\alpha \notin j_{W_{0,\alpha}}(A \cap \alpha)$.

Remark 6.2. To get this set $A \in W_1$ it suffices to taking any $A \in U_1 \setminus U_0$ and intersect it with the collection of all $\alpha < \kappa$ for which (1) holds.

Let $\mathbb{G}_\kappa$ be the Easton-supported iteration defined recursively as follows. The iteration just forces non-trivially at measurables $\alpha \in A$. Suppose that $\mathbb{G}_\alpha$ has been defined. If α is a successor point of A then $|\mathbb{G}_\alpha| < \alpha$ and $W_{0,\alpha}$ lifts naturally to a $V^{\mathbb{G}_\alpha}$-measure $\overline{W}_{0,\alpha}$. In that case the αth-stage of the iteration is declared to be $\mathbb{T}_{\overline{W}_{0,\alpha}}$. Alternatively, suppose that α is a limit point of A. Once again one can lift $W_{0,\alpha}$ to a $V^{\mathbb{G}_\alpha}$-measure $\overline{W}_{0,\alpha}$ as follows:

$$(\dot X_\beta)_{G_\alpha} \in \overline{W}_{0,\alpha} :\Longleftrightarrow \exists p \in G_\alpha \ (p^\frown p_\beta \Vdash^{M_{W_{\alpha,0}}}_{j_{W_{\alpha,0}}(\mathbb{G}_\alpha)} [\mathrm{id}]_{W_{0,\alpha}} \in j_{W_{\alpha,0}}(\dot X_\beta)),$$

where $\langle p_\beta \mid \beta < \alpha^+\rangle$ is a $\leq^*$-increasing sequence in $j_{W_{\alpha,0}}(\mathbb{G}_\alpha)/G_\alpha$ with:[9]

(i) p_β decides the sentence "$[\mathrm{id}]_{W_{\alpha,0}} \in j_{W_{\alpha,0}}(\dot X_\beta)$" where $\langle \dot X_\beta \mid \beta < \alpha^+\rangle$ is an enumeration of all $\mathbb{G}_\alpha$-names for subsets of α.

(ii) $\langle p_\beta \mid \beta < \alpha^+\rangle$ is chosen to be minimal with respect to some well-ordering of a big enough fragment of V (see [Git86, §2] for details).

Finally we declare the αth-stage of the iteration to be $\mathbb{T}_{\overline{W}_{0,\alpha}}$.

The above yields the preparatory Gitik's iteration $\mathbb{G}_\kappa$. Let $G \subseteq \mathbb{G}_\kappa$ be V-generic and let us extend the V-measures W_0 and W_1 to measures $\overline{W}_0$ and $\overline{W}_1$ in $V[G]$. Once these measures $\overline{W}_0$ and $\overline{W}_1$ are obtained we shall define (in $V[G]$) a poset $\mathbb{P}(\kappa,2)$ such that forcing over $V[G]$ produces:

(1) An ω^2-sequence $\langle \kappa_\alpha \mid \alpha < \omega^2\rangle$ converging to κ;
(2) A $V[G]$-generic function for $\mathrm{Add}(\kappa,1)^{V[G]}$.

First, since $A \notin W_0$ we have that $\kappa \notin j_{W_0}(A)$ so, as before, W_0 extends to $\overline{W}_0$. Second let us show how to lift W_1 to $\overline{W}_1$. For this let us fix $\pi\colon \kappa \to \kappa$ such that $j_{\overline{W}_0}(\pi)([\mathrm{id}]_{\overline{W}_0}) = \kappa$.

[9]The key point to obtain such an $\overline{W}_{0,\alpha}$ is Clause (2) above. Indeed, thanks to this one has that $j_{W_{0,\alpha}}(\mathbb{G}_\alpha)$ factors as a two-step iteration $\mathbb{G}_\alpha * \mathbb{G}_{tail}$, where the latter is an α^+-closed iteration with respect to the corresponding Prikry order $\leq^*$.

Definition 6.3. A sequence of ordinals $\langle \alpha_0, \dots, \alpha_n \rangle \in [\kappa]^{<\omega}$ is called π-*increasing* if $\alpha_i < \pi(\alpha_{i+1})$ for all $i < n$.

Definition 6.4. For each π-increasing sequence $t \in [\kappa]^{<\omega}$ define

$$\overline{W}_1(t) := \{(\dot{X}_\alpha)_G \mid \exists p \in G\, \exists \dot{T}\ (p^\frown\{\langle t, \dot{T}\rangle\}^\frown p_\alpha \Vdash^{M_{W_1}}_{j_{W_1}(\mathbb{G}_\kappa)} \kappa \in j_{W_1}(\dot{X}_\alpha))\},$$

where $\langle \dot{X}_\alpha \mid \alpha < \kappa^+\rangle$ and $\langle p_\alpha \mid \alpha < \kappa^+\rangle$ are as in [Git86, §3].

Remark 6.5. By our inductive construction the κth-stage of the iteration $j_{W_1}(\mathbb{G}_\kappa)$ is exactly the Tree Prikry forcing $\mathbb{T}_{\overline{W}_0}$. For this one has to argue that the lifting of the measure W_0 is the same both when computed in $V[G]$ and in $M_{W_1}[G]$. This is where the well-ordering of the universe plays an essential role. We defer to provide further details about this aspect and instead refer our readers to [Git86, Lemma 2.1].

It is not hard to check that $\overline{W}_1(t)$ is a measure in $V[G]$ concentrating on

$$\{\alpha < \kappa \mid \text{“The Prikry sequence } b_\alpha \text{ for } \mathbb{T}_{\overline{W}_{0,\alpha}} \text{ over } V[G_\alpha] \text{ end-extends } t\text{”}\}.$$

In addition the following properties hold upon $\overline{W}_1(t)$:

(1) $\overline{W}_1(t)$ is not normal as it concentrates on singular cardinals.
(2) Since $\mathbb{G}_\kappa$ is κ-cc, W_1 is normal and $W_1 \subseteq \overline{W}_1(t)$,

$$(\mathrm{Cub}_\kappa)^{V[G]} \subseteq \overline{W}_1(t).$$

Using $\overline{W}_0$ and $\langle \overline{W}_1(t) \mid t \in [\kappa]^{<\omega} \wedge t \text{ is } \pi\text{-increasing}\rangle$ we present Gitik's forcing $\mathbb{P}(\kappa, 2)$ adding an ω^2-sequence to κ without adding bounded sets.

Definition 6.6. A sequence $t = \langle \xi_0, \dots, \xi_k\rangle \in [\alpha]^{<\omega}$ is 2-*coherent* if

(1) t is increasing;
(2) $o^{\vec{U}}(\xi_i) \leq 1$ for all $i < k$;
(3) for all $i < k$ let $i^* \leq i$ be the first index such that

$$o^{\vec{U}}(\xi_j) < o^{\vec{U}}(\xi_i) \text{ for all } i^* \leq j < i.$$

Then, b_{ξ_i} end-extends $\bigcup_{i^* \leq j < i}(b_{\xi_j} \cup \{\xi_j\})$ where each b_{ξ_ℓ} denotes the generic sequence added by $\mathbb{T}_{\overline{W}_{\xi_\ell}}$ over $V^{\mathbb{G}_{\xi_\ell}}$.

Given a 2-coherent sequence t we denote

$$b_t := \bigcup_{\xi \in r} b_\xi.$$

Also we denote by $t \restriction 1$ the following sequence: If $o^{\vec{U}}(\max(t)) = 1$ then $t \restriction \bar{\beta} := \varnothing$. Otherwise, let $i^* < |t|$ be the first index with $o^{\vec{U}}(\xi_j) = 0$ for all $i^* \leq j < i$ and set $t \restriction \bar{\beta} := \langle \xi_{i^*}, \dots, \xi_{|t|-1}\rangle$.

Definition 6.7. A condition in $\mathbb{P}(\alpha, 2)$ is a pair $\langle t, T\rangle$ where:

(1) t is 2-coherent;
(2) T is a tree on $[\kappa]^{<\omega}$ with trunk $\varnothing \in T$;

(3) $t^\frown s$ is 2-coherent for all $s \in T$, $\mathrm{Succ}_T(s) = \bigcup_{\bar\beta<2} \mathrm{Succ}_{T,\bar\beta}(s)$ and

$$\mathrm{Succ}_{T,0}(s) \in \overline{W}_0 \ \wedge\ \mathrm{Succ}_{T,1}(s) \in \overline{W}_1((t^\frown s) \restriction 1).$$

Given $\langle t, T\rangle, \langle s, S\rangle \in \mathbb{P}(\alpha, 2)$ write $\langle s, S\rangle \leq^* \langle t, T\rangle$ iff $t = s$ and $T \subseteq S$. Also, say that $\langle t, T\rangle$ and $\langle s, S\rangle$ are *equivalent* if $b_t = b_s$ and $T = S$.

Let $H \subseteq \mathbb{P}(\kappa, 2)$ a generic filter over $V[G]$. Let C_H be the ω^2-sequence added by H and $\langle \kappa_n \mid n < \omega\rangle$ be the increasing enumeration of the limit points of C_H (see Definition 6.21). Then

$$C_H = \bigcup_{n<\omega} b_{\kappa_n} \cup \{\kappa_n\}.$$

For each $n < \omega$ the Tree Prikry generic b_{κ_n} for $\mathbb{T}_{\overline{W}_{0,\kappa_n}}$ (over $V[G_{\kappa_n}]$) is, by the Mathias criterion for the Tree Prikry forcing [Ben19], V-generic for $\mathbb{T}_{W_{0,\kappa_n}}$. Thus, by our Clause (1) in page 28, this generates a V-generic Cohen function $f^*_{\kappa_n}$ for $\mathrm{Add}(\kappa_n, 1)^V$.

Lemma 6.8. *$f^*_{\kappa_n}$ induces a $V[G_{\kappa_n}]$-generic Cohen for $\mathrm{Add}(\kappa_n, 1)^{V[G_{\kappa_n}]}$.*

Proof. Since $\mathbb{G}_{\kappa_n}$ is an κ_n-cc forcing of size κ_n, the poset $\mathrm{Add}(\kappa_n, 1)^V$ is isomorphic to the term-space forcing $\mathbb{A}(\mathbb{G}_\alpha, \dot{\mathrm{Add}}(\alpha, 1))$ (see [Cum92a, p.9]). Thus, modulo isomorphisms, $f^*_{\kappa_n}$ is V-generic for this latter poset. By standard arguments about the term space forcing (see e.g. [Cum10, Proposition 22.3]), $f^*_{\kappa_n}$ and G_{κ_n} together induce a $V[G_{\kappa_n}]$-generic filter for $\mathrm{Add}(\kappa_n, 1)^{V[G_{\kappa_n}]}$. This completes the verification of the lemma. □

For simplicity, let us keep calling $f^*_{\kappa_n}$ the generic for $\mathrm{Add}(\kappa_n, 1)^{V[G_{\kappa_n}]}$.

Lemma 6.9. *$f^*_\kappa := \bigcup_{n<\omega} f^*_{\kappa_n} \restriction [\kappa_{n-1}, \kappa_n)$ is $V[G]$-generic for $\mathrm{Add}(\kappa, 1)^{V[G]}$.*

Proof. Let $\mathcal{A} \in V[G]$ be a maximal antichain for $\mathrm{Add}(\kappa, 1)^{V[G]}$. Consider the function $f\colon \kappa \to \kappa$ defined in $V[G]$ as follows. For each $p \in \mathrm{Add}(\alpha, 1)^{V[G]}$ let $\beta(p) < \kappa$ be the least ordinal for which there is $q_p \in \mathcal{A} \cap \mathrm{Add}(\beta(p), 1)^{V[G]}$ compatible with p. Set $f(\alpha) := \sup_{p\in \mathrm{Add}(\alpha,1)^{V[G]}} \beta(p)$.

Let C be the club of closure points of f. Since $C, A \in \bigcap_{t\in[\kappa]^{<\omega}} \overline{W}_1(t)$ it follows that $\langle \kappa_n \mid n \geq n_0\rangle \subseteq A \cap C$ for some $n_0 < \omega$. Let κ_n be one of such ordinals. Note that $\mathcal{A} \cap \mathrm{Add}(\kappa_n, 1)^{V[G]} = \mathcal{A} \cap \mathrm{Add}(\kappa_n, 1)^{V[G_{\kappa_n}]}$ is a maximal antichain for $\mathrm{Add}(\kappa_n, 1)^{V[G_{\kappa_n}]}$. By further shrinking $A \cap C$ we may assume (as the next claim demonstrates) that $\mathcal{A} \cap \mathrm{Add}(\kappa_n, 1)^{V[G_{\kappa_n}]} \in V[G_{\kappa_n}]$:

Claim 6.10. *$\{\alpha < \kappa \mid \mathcal{A} \cap \mathrm{Add}(\alpha, 1)^{V[G_\alpha]} \in V[G_\alpha]\} \in \overline{W}_1(t)$ for all t.*

Proof of claim. Fix an arbitrary π-increasing sequence t. Fix $\dot{\mathcal{A}}$ and $\dot X$, $\mathbb{G}_\kappa$-names for $\mathcal{A}$ and the above-displayed set, respectively. Let $p \in G$ forcing the above properties about $\dot{\mathcal{A}}$ and $\dot X$. We can moreover assume that $\dot{\mathcal{A}} \subseteq V_\kappa$ because $\mathbb{G}_\kappa$ is κ-cc and $\mathbb{G}_\kappa \subseteq V_\kappa$. In particular, $j_{W_1}(\dot{\mathcal{A}}) \cap V_\kappa = \dot{\mathcal{A}} \in M_{W_1}$ and thus $\dot{\mathcal{A}}_G \in M_{W_1}[G]$. Note that this is still true in any generic extension of $M_{W_1}[G]$ by the tail forcing $j_{W_1}(\mathbb{G}_\kappa)/G$. Therefore, there are $p \le q \in G$, $\langle t, \dot{T}\rangle \in \mathbb{P}(\kappa, 2)$ and p_α with

$$q \cup \{\langle t, \dot{T}\rangle\} \cup p_\alpha \Vdash_{j_{W_1}(\mathbb{G}_\kappa)} j_{W_1}(\dot{\mathcal{A}}) \cap \mathrm{Add}(\kappa, 1)^{V[\dot{G}]} \in M_{W_1}[\dot{G}].$$

Since $j_{W_1}(q) = q$ forces the same relationship between $j_{W_1}(\dot{\mathcal{A}})$ and $j_{W_1}(\dot{X})$, the above shows that $q \cup \{\langle t, \dot{T}\rangle\} \cup p_\alpha$ forces "$\kappa \in j_{W_1}(\dot{X})$". By definition, this is the same as saying that $\dot{X}_G \in \overline{W}_1(t)$. $\square$

So, $\mathcal{A}$ must include a restriction of the function $f^* \restriction \kappa_n$ in that this is a bounded modification of $f^*_{\kappa_n}$, which was generic over $V[G_{\kappa_n}]$. Thus, the antichain $\mathcal{A}$ intersects f^* and we are done. $\square$

6.2. **Adding a Cohen function along an ω_1-sequence.** As in the previous section our ground model will be denoted by V_0 and we shall assume that both the GCH holds and that the model accomodates a Mitchell-increasing sequence $\langle U_i \mid i < \omega_1\rangle$ of normal measures over κ. Again, we perform the same forcing preparation $\mathbb{P}_\kappa$ of §6.1 based on the lottery sum of the trivial forcing and $\mathrm{Add}(\alpha, 1)$ for all inaccessibles $\alpha < \kappa$.

Let $G \subseteq \mathbb{P}_\kappa$ be V_0-generic.

Lemma 6.11. *In $V_0[G]$, U_i extends to a κ-complete ultrafilter W_i such that:*
(1) $\langle W_i \mid i < \omega_1\rangle$ is Mitchell increasing;
(2) W_i is normal except whenever $i = 0$;
(3) $(\mathrm{Cub}_\kappa)^{V_0[G]} \subseteq W_0$ and W_0 is such that forcing with the Tree-Prikry forcing $\mathbb{T}_{W_0}$ over $V_0[G]$ introduces an $\mathrm{Add}(\kappa, 1)^{V_0[G]}$-generic.

Proof. We define a sequence a generics $\langle G_i \mid i < \omega_1\rangle$ so that $G_i \restriction \kappa = G$ and

$$G_i \in M_{U_{i+1}}[G] \text{ is } (M_{U_i})^{M_{U_{i+1}}}\text{-generic for } j_{U_i}(\mathbb{P}_\kappa).$$

The point is the following: from the perspective of $M_{U_{i+1}}[G]$, $j_{U_i}(\mathbb{P}_\kappa)/G$ is a forcing of cardinality κ^+ and there are only κ^+-many maximal antichains to meet. In addition, by the usual arguments involving the commutative diagram between j_{U_i} and $j_{U_{i+1}}$, both M_{U_i} and $(M_{U_i})^{M_{U_{i+1}}}$ agree on $(V_0)_{j_{U_i}(\kappa)+1}$ and therefore G_i is M_{U_i}-generic. Note that $G_i \in M_{U_j}[G]$ for all $i < j$.

For each $0 < i < \omega_1$ lift $j_{U_i} \subseteq j^*_i : V_0[G] \to M_{U_i}[G_i]$ and let $W_i \in V_0[G]$ be the lifted measure. Clearly, $M_{W_i} = M_{U_i}[G_i]$. For $i = 0$, we lift the second iteration $j_{U_0^2} \subseteq j_{W_0} : V[G] \to M_{U^2}[G_0]$ so that W_0 concentrate on Cohens. Namely, in the case where $i = 0$ the measure W_0 will be non-normal, yet it will satisfy the blanket assumptions described in Setup 1 of page 27.

Claim 6.12. *$W_i \in M_{W_\ell}$ for all $\ell > i$.*

Proof of claim. Let us first assume that $\ell > i$, and consider the standard commutative diagram between the measures U_ℓ and U_i; namely,

$$\begin{array}{ccc} V_0 & \xrightarrow{j_{U_\ell}} & M_{U_\ell} \\ \Big\downarrow{\scriptstyle j_{U_i}} & & \Big\downarrow{\scriptstyle (j_{U_i})^{M_{U_\ell}}} \\ M_{U_i} & \xrightarrow{j_{U_i}(U_\ell)} & N, \end{array}$$

where $(j_{U_i})^{M_\ell}$ stands for the ultrapower embedding by U_i over M_{U_ℓ}.

Since W_i is the lifting of j_{U_i} by the poset $j_{U_i}(\mathbb{P}_\kappa)$, $X \in W_i$ if and only if there is a $\mathbb{P}_\kappa$-name $\dot{X}$ for a subset of κ such that $\dot{X}_{G_i} = X$ and

$$p \Vdash^{M_{U_i}} [\mathrm{id}]_{W_i} \in j_{U_i}(\dot{X})$$

for some $p \in G_i$. Thus, note that W_i is definable via G_i and j_{U_i}.

On the one hand, $G_i \in M_{U_\ell}[G] \subseteq M_{W_\ell}$. On the other hand, by κ-ccness of $\mathbb{P}_\kappa$ every $\mathbb{P}_\kappa$-name $\dot{X}$ for a subset of κ can be assumed to be a member of $(V_0)_{\kappa+1}$. We shall next show that

$$(j_{U_i})^{M_{U_\ell}} \restriction (V_0)_{\kappa+1} = j_{U_i} \restriction (V_0)_{\kappa+1}$$

which combined with our previous comments will establish $W_i \in M_{W_\ell}$.

To simplify notations, let us denote $j_{\ell,i} := (j_{U_i})^{M_{U_\ell}}$ and $j_{i,\ell} := j_{U_i}(U_\ell)$. Fix $P \subseteq (V_0)_\kappa$. Since $\mathrm{crit}(j_{i,\ell}) = j_{U_i}(\kappa)$ we have

$$j_{U_i}(P) = j_{i,\ell}(j_{U_i}(P)) \cap (M_{U_i})_{j_{U_i}(\kappa)} = j_{i,\ell}(j_{U_i}(P)) \cap N_{j_{U_i}(\kappa)}.$$

By commutativity of the diagram this amounts to saying

$$j_{U_i}(P) = j_{\ell,i}(j_{U_\ell}(P)) \cap N_{j_{U_i}(\kappa)} = j_{\ell,i}(j_{U_\ell}(P) \cap (M_{U_\ell})_\kappa).$$

Once again, since $\mathrm{crit}(j_{U_\ell}) = \kappa$, $(M_{U_\ell})_\kappa = (V_0)_\kappa$ and

$$j_{U_\ell}(P) \cap (M_{U_\ell})_\kappa = j_{U_\ell}(P) \cap (V_0)_\kappa = P.$$

Combining these two latter equations we obtain

$$j_{U_i}(P) = j_{\ell,i}(P),$$

as needed.

The proof for $i = 0$ is identical, bearing in mind that if $U_0 \in M_{U_\ell}$ then also $U_0^2 \in M_{U_\ell}$ and thus the argument for W_0 and W_ℓ is the same as the one of the previous paragraph, working with the commutative diagram of U_0^2 and U_ℓ. □

The above completes the proof of the lemma. □

Setup 2. We denote our new ground model by V. Invoking Corollary 2.6 inside V we derive an almost coherent sequence $\vec{U}$ on κ of length ω_1 such that $U(\kappa, i) = W_i$ (see Definition 2.5). By Changing $\vec{U}$ on a null-set, we may assume that for every measurable cardinal $\alpha < \kappa$, $U(\alpha, 0)$ is a non-normal α-complete ultrafilter witnessing the clauses provided in Setup 1.

Definition 6.13. For each $i < \omega_1$ define

$$\mathrm{dom}_1(\vec{U}) := \{\eta \le \kappa \mid o^{\vec{U}}(\eta) > 0\}.$$

As in the previous section we begin defining an Easton-supported iteration

$$\mathbb{G}_\kappa := \varinjlim \langle \mathbb{G}_\alpha; \dot{\mathbb{Q}}_\alpha \mid \alpha < \kappa\rangle$$

using the almost coherent sequence $\vec{U} := \langle \vec{U}(\alpha,\beta) \mid \alpha \le \kappa,\, \beta < o^{\vec{U}}(\alpha)\rangle$.

Fix $\alpha \le \kappa$ and suppose that $\mathbb{G}_\alpha$ has been defined. If $\alpha \notin \mathrm{dom}_1(\vec{U})$ we declare $\mathbb{Q}_\alpha$ to be the trivial forcing. Otherwise, $\alpha \in \mathrm{dom}_1(\vec{U})$ and we have two options: either $\mathrm{dom}_1(\vec{U}) \cap \alpha$ is bounded in α or it is not. In the former case, $o^{\vec{U}}(\alpha) = 1$ and by standard arguments due to Lévy and Solovay, $U(\alpha, 0)$ extends to a measure $\bar{U}(\alpha, 0)$ in $V^{\mathbb{P}_\alpha}$. In this latter case we declare $\mathbb{Q}_\alpha$ to be $\mathbb{T}_{\bar{U}(\alpha,0)}$, the Tree Prikry forcing relative to $\bar{U}(\alpha, 0)$.

So, suppose that $\mathrm{dom}_1(\vec{U}) \cap \alpha$ is unbounded in α. We define sequences

$$\langle \mathbb{P}(\alpha,\beta) \mid \beta \le o^{\vec{U}}(\alpha)\rangle \text{ and } \langle U(\alpha,\beta,t) \mid \beta < o^{\vec{U}}(\alpha),\, t \in \mathcal{C}_{\alpha,\beta}\rangle$$

as follows. Let $\mathbb{P}(\alpha, 0)$ be the trivial forcing and $U(\alpha, 0, \varnothing)$ the measure defined as follows. Let $j^\alpha_0 \colon V \to N^\alpha_0$ be the ultrapower embedding by $U(\alpha, 0)$. By coherency, $j^\alpha_0(\vec{U}) \restriction \alpha + 1 = \vec{U} \restriction (\alpha, 0) = \varnothing$. Hence, $j^\alpha_0(\mathbb{G}_\alpha)$ factors as

$$\mathbb{G}_\alpha * \{\varnothing\} * \mathbb{G}_{(\alpha, j^\alpha_0(\alpha))}.$$

The tail forcing $\mathbb{G}_{(\alpha,j^\alpha_0(\alpha))}$ has the Prikry property and is α^{++}-closed with respect to the $\le^*$-order. Standard arguments allow us to produce an extension $U(\alpha, 0, \varnothing)$ of $U(\alpha, 0)$ in $V^{\mathbb{G}_\alpha}$. Note that $U(\alpha, 0, \varnothing)$ extends the club filter Cub_α as computed in $V^{\mathbb{G}_\alpha}$: Indeed, $U(\alpha, 0)$ extends the V-club filter and $\mathbb{P}_\alpha$ is α-cc (so every $V^{\mathbb{G}_\alpha}$-club contains a V-club). To make the forthcoming construction work smoothly we follow Gitik's ideas [Git86, §3] and define $U(\alpha, 0, \varnothing)$ relative to a fix well-ordering of a large-enough fragment of the set-theoretic universe. More precisely, we define $U(\alpha, 0, \varnothing)$ analogously to $\overline{W}_{\alpha,0}$ in page 28.

Suppose that both $\mathbb{P}(\alpha, \bar{\beta})$ and $U(\alpha, \bar{\beta}, t)$ have been constructed for all $\bar{\beta} < \beta \le o^{\vec{U}}(\alpha)$ in $V^{\mathbb{G}_\alpha}$. To proceed we need the notion of β-coherency:

Definition 6.14. A sequence $t = \langle \xi_0, \dots, \xi_k\rangle \in [\alpha]^{<\omega}$ is β-coherent if

(1) t is increasing;
(2) $o^{\vec{U}}(\xi_i) < \beta$ for all $i < k$;
(3) for all $i < k$ let $i^* \le i$ be the first index such that $o^{\vec{U}}(\xi_j) < o^{\vec{U}}(\xi_i)$ for all $i^* \le j < i$. Then, b_{ξ_i} end-extends $\bigcup_{i^* \le j < i}(b_{\xi_j} \cup \{\xi_j\})$. Where b_{ξ_j} is the generic sequence added by $\mathbb{P}(\xi_j, o(\xi_j))$ over $V^{\mathbb{G}_{\xi_j}}$.

Denote by $\mathcal{C}_{\alpha,\beta}$ the collection of all β-coherent sequences in $[\alpha]^{<\omega}$. Given $t, s \in \mathcal{C}_{\alpha,\beta}$ we say that t and s are *equivalent* if $b_t = b_s$ where

$$b_r := \textstyle\bigcup_{\xi \in r} b_\xi \text{ for } r \in \{t, s\}.$$

For each $\bar{\beta} < \beta$ denote by $t \upharpoonright \bar{\beta}$ the following sequence: If $o^{\vec{U}}(\max(t)) \geq \bar{\beta}$ then $t \upharpoonright \bar{\beta} := \varnothing$. Otherwise, let $i^* < |t|$ be the first index with $o^{\vec{U}}(\xi_j) < \bar{\beta}$ for all $i^* \leq j < i$ and set $t \upharpoonright \bar{\beta} := \langle \xi_{i^*}, \dots, \xi_{|t|-1} \rangle$.

We can now define the poset $\mathbb{P}(\alpha, \beta)$:

Definition 6.15. A condition in $\mathbb{P}(\alpha, \beta)$ is a pair $\langle t, T \rangle$ where:

(1) $t \in \mathcal{C}_{\alpha,\beta}$;
(2) T is a tree on $[\alpha]^{<\omega}$ with trunk $\varnothing \in T$;
(3) $t^\frown s$ is β-coherent for all $s \in T$, $\mathrm{Succ}_T(s) = \bigcup_{\bar{\beta} < \beta} \mathrm{Succ}_{T,\bar{\beta}}(s)$ and

$$\mathrm{Succ}_{T,\bar{\beta}}(s) \in U(\alpha, \bar{\beta}, (t^\frown s) \upharpoonright \bar{\beta}).$$

Given $\langle t, T \rangle, \langle s, S \rangle \in \mathbb{P}(\alpha, \beta)$ write $\langle s, S \rangle \leq^* \langle t, T \rangle$ iff $t = s$ and $T \subseteq S$.
Also, say that $\langle t, T \rangle$ and $\langle s, S \rangle$ are *equivalent* if $b_t = b_s$ and $T = S$.

Remark 6.16. Note that, formally speaking, $\mathrm{Succ}_{T,\bar{\beta}}(\cdot)$ depends also on the entire condition $\langle t, T \rangle$. To avoid overcomplicated notations we shall keep denoting the set of successors in that way, in place of $\mathrm{Succ}_{\langle t,T \rangle,\bar{\beta}}(\cdot)$.

Definition 6.17 (Minimal extensions)**.** For $\langle t, T \rangle \in \mathbb{P}(\alpha, \beta)$ and $\langle \nu \rangle \in T$,

$$\langle t, T \rangle^\frown \langle \nu \rangle := \langle t^\frown \langle \nu \rangle, T_{\langle \nu \rangle} \setminus V_{\nu+1} \rangle.$$

As customary, $T_{\langle \nu \rangle} := \{ s \in T \mid \langle \nu \rangle^\frown s \in T \}$.
In general for $\vec{\nu} \in T$ define $\langle t, T \rangle^\frown \vec{\nu}$ by recursion on the length of $\vec{\nu}$.

The standard order of $\mathbb{P}(\alpha, \beta)$ is defined as a combination of $\leq^*$ and ${}^\frown \vec{\nu}$:

Definition 6.18. For $\langle t, T \rangle, \langle s, S \rangle \in \mathbb{P}(\alpha, \beta)$ write $\langle t, T \rangle \leq \langle s, S \rangle$ if and only if there is $\vec{\nu} \in S$ such that $\langle t, T \rangle$ is equivalent to a $\leq^*$-extension of $\langle s, S \rangle^\frown \vec{\nu}$.

Remark 6.19. If $\langle t, T \rangle$ and $\langle s, S \rangle$ are equivalent then $\langle t, T \rangle \leq \langle s, S \rangle$ and $\langle s, S \rangle \leq \langle s, S \rangle$. Thus both conditions force the same information.

Next, we define the measures $\langle U(\alpha, \beta, t) \mid t \in \mathcal{C}_{\alpha,\beta} \rangle$ as follows:

Definition 6.20. For each $t \in \mathcal{C}_{\alpha,\beta}$, define

$$U(\alpha, \beta, t) := \{ (\dot{X}_\alpha)_{G_\alpha} \mid \exists p \in G_\alpha \, \exists \dot{T} \, (p^\frown \{ \langle t, \dot{T} \rangle \}^\frown p_\gamma \Vdash_{j^\alpha_\beta(\mathbb{G}_\alpha)} \alpha \in j^\alpha_\beta(\dot{X}_\gamma)) \},$$

where $\langle \dot{X}_\gamma \mid \gamma < \alpha^+ \rangle$ and $\langle p_\gamma \mid \gamma < \alpha^+ \rangle$ are as in [Git86, §3] and j^α_β denotes the ultrapower embedding by $U(\alpha, \beta)$.

The above completes the inductive definition of

$$\mathbb{P}(\alpha, \beta) \text{ and } \langle U(\alpha, \beta, t) \mid t \in \mathcal{C}_{\alpha,\beta} \rangle$$

for all $\alpha \leq \kappa$ and $\beta \leq o^{\vec{U}}(\alpha)$. Finally let $\dot{\mathbb{Q}}_\alpha$ a $\mathbb{G}_\alpha$-name for $\mathbb{P}(\alpha, o^{\vec{U}}(\alpha))$.

Gitik showed that $\langle \mathbb{P}(\alpha, o^{\vec{U}}(\alpha)), \leq, \leq^* \rangle$ is a Prikry-type forcing [Git86, Lemma 3.11]. It is also easy to show that $\mathbb{P}(\alpha, o^{\vec{U}}(\alpha))$ is α^+-cc and that $\langle \mathbb{P}(\alpha, o^{\vec{U}}(\alpha)), \leq^* \rangle$ is an α^+-closed forcing. Thus, forcing with $\mathbb{P}(\alpha, o^{\vec{U}}(\alpha))$

does not collapse cardinals. However, forcing with $\mathbb{P}(\alpha, o^{\vec{U}}(\alpha))$ adds a cofinal sequence to α with order-type $\omega^{o^{\vec{U}}(\alpha)}$. As a result this forcing changes the cofinality of α – details are provided below.

Definition 6.21. Let $H \subseteq \mathbb{P}(\alpha, o^{\vec{U}}(\alpha))$ a $V[G_\alpha]$-generic. Define

$$b_\alpha := \bigcup\{b_\beta \mid \exists\langle t, T\rangle \in H,\ \beta \in t\}.$$

Note that if $\langle t, T\rangle \leq \langle s, S\rangle$ then s is equivalent to an initial segment of t and therefore b_t end-extends b_s. It follows that for each $\langle t, T\rangle \in H$, b_α end-extends b_t. Arguing inductively, one can now prove that b_α is a club with $\text{otp}(b_\alpha) = \omega^{o^{\vec{U}}(\alpha)}$. It follows that the cofinality of α in $V[G_\alpha]$ is determined by this order-type, and in particular we have the following:

Corollary 6.22. *Let G_κ be V-generic for $\mathbb{P}_\kappa$ and let G be $V^* = V[G_\kappa]$-generic for $\mathbb{P}(\kappa, \omega_1)$. Then $\text{cf}^{V^*[G]}(\kappa) = \omega_1$.*

Let G_κ be V-generic for $\mathbb{P}_\kappa$ and let $V^* = V[G_\kappa]$. By definition of the iteration $\mathbb{P}_\kappa$, for every $\alpha \in \text{dom}_1(\vec{U})$, we have a $V[G_\alpha]$-generic sequence b_α for $\mathbb{P}(\alpha, o^{\vec{U}}(\alpha))$. Note that every $\xi \in b_\alpha$ with $\vec{U}$-order 0 is in some b_γ for $\gamma \leq \alpha$ with $o^{\vec{U}}(\gamma) = 1$, and by definition, b_γ is generic for $\mathbb{T}_{\bar{U}(\alpha,0)}$. It follows that f_ξ (the V_0-generic functions for $\text{Add}(\xi, 1)$) is defined. In particular, we may assume that if $C_G = \langle \kappa_i \mid i < \omega_1\rangle$ is a V^*-generic filter for $\mathbb{P}(\kappa, \omega_1)$, then for every $i < \omega_1$, $\kappa_{i+1} \in Y_0$. And in particular $f_{\kappa_{i+1}} : \kappa_{i+1} \to \kappa_{i+1}$.

Theorem 6.23. *Let $G \subseteq \mathbb{P}(\kappa, \omega_1)$ be V^*-generic and let $C_G = \langle \kappa_i \mid i < \omega_1\rangle$ be the generic club sequence. Then,*

$$f^* := f_0 \restriction \kappa_0 \cup \textstyle\bigcup_{i<\omega_1} f_{\kappa_{i+1}} \restriction [\kappa_i, \kappa_{i+1})$$

is V^*-generic for $\text{Add}(\kappa, 1)^{V^*}$.

Proof. Let us denote by $\text{Succ}(C_G)$ the increasing sequence of successor points of C_G; namely $\langle \kappa_{i+1} \mid i < \omega_1\rangle$. For each $\alpha \in C_G \cup \{\kappa\}$ define

$$f^*_\alpha := f_0 \restriction \kappa_0 \cup \textstyle\bigcup_{\beta \in \text{Succ}(C_G) \cap \alpha} f_\beta \restriction [\beta^-, \beta)$$

where β^- stands for the predecessor of β in C_G.

We will show that for every $\alpha \in C_G \cup \{\kappa\}$, f^*_α is $V[G_\alpha]$-generic for $\text{Add}(\alpha, 1)^{V[G_\alpha]}$. In particular, $f^* = f^*_\kappa$ will be generic over $V^* = V[G_\kappa]$.

The proof is by induction on $0 < \gamma \leq \omega_1$, and the induction step is proved for all $\alpha \in C_G \cup \{\kappa\}$ with $o^{\vec{U}}(\alpha) = \gamma$. For $o^{\vec{U}}(\alpha) = 1$, $\mathbb{P}(\alpha, 1)$ is just the Tree-Prikry forcing with $U(\alpha, 0, \varnothing) \supseteq U(\alpha, 0)$. In this case note that

$$f^*_\alpha = f^*_{\alpha^*} \cup \bigcup_{i<\omega} f_{\alpha_i} \restriction [\alpha_{i-1}, \alpha_i),$$

where $\langle \alpha_i \mid i < \omega\rangle$ is the Prikry sequence b_α added by $\mathbb{P}(\alpha, 1)$, and $\alpha^* \in C_G \cap \alpha$ is the last ordinal such that $o^{\vec{U}}(\alpha^*) \geq 0^{\vec{U}}(\alpha)$ and $o^{\vec{U}}(\beta) < o^{\vec{U}}(\alpha)$

for all $\beta \in C_G \cap (\alpha^*, \alpha)$. The sequence $\langle \alpha_n \mid n < \omega \rangle$ is also V_0-generic for $U(\alpha, 0)$ (by the Mathias criterion) and by the construction of $U(\alpha, 0)$,

$$\bigcup_{n<\omega} f_{\alpha_n} \restriction [\alpha_{n-1}, \alpha_n)$$

is $\mathrm{Add}(\alpha, 1)^{V_0[G\restriction\alpha]}$-generic (see Setup 2). Since V is a generic extension of $V_0[G \restriction \alpha]$ by an α^+-closed forcing (namely, the tail of the preliminary lottery iteration), this function is also generic for $\mathrm{Add}(\alpha, 1)^V$.

Claim 6.24. $\bigcup_{n<\omega} f_{\alpha_n} \restriction [\alpha_{n-1}, \alpha_n)$ *is a* $V[G_\alpha]$*-generic for* $\mathrm{Add}(\alpha, 1)^{V[G_\alpha]}$. *In particular,* f^*_α *is* $V[G_\alpha]$*-generic for* $\mathrm{Add}(\alpha, 1)^{V[G_\alpha]}$.

Proof of Claim. First we note that $V[G_\alpha]$ is a forcing extension of V by $\mathbb{G}_\alpha$, which is an α-c.c forcing of size α. It follows that $\mathrm{Add}(\alpha, 1)^V$ is isomorphic to the term-space forcing $\mathbb{A}(\mathbb{G}_\alpha, \dot{\mathrm{Add}}(\alpha, 1))$ (see [Cum92a, p.9]). Thus, modulo isomorphism, $\bigcup_{n<\omega} f_{\alpha_n} \restriction [\alpha_{n-1}, \alpha_n)$ is V-generic for this latter poset. By standard arguments about the term space forcing, $\bigcup_{n<\omega} f_{\alpha_n} \restriction [\alpha_{n-1}, \alpha_n)$ and G_α together induce a $V[G_\alpha]$-generic filter for $\mathrm{Add}(\alpha, 1)^{V[G_\alpha]}$.

For the last claim, f^*_α is a bounded modification of $\bigcup_{n<\omega} f_{\alpha_n} \restriction [\alpha_{n-1}, \alpha_n)$ using a function in $V[G_\alpha]$ (i.e. $f^*_{\alpha^*}$) and so f^*_α is also $V[G_\alpha]$-generic. □

Let us argue for the general case. Our induction hypothesis is

$$\forall \beta \in C_G \cap \alpha (o^{\vec{U}}(\beta) < o^{\vec{U}}(\alpha) \Rightarrow f^*_\beta \text{ is } \mathrm{Add}(\beta, 1)^{V[G_\beta]}\text{-generic over } V[G_\beta]).$$

Claim 6.25. f^*_α *is* $V[G_\alpha]$*-generic.*

Proof of claim. Let $\mathcal{A} \in V[G_\alpha]$ be a maximal antichain for $\mathrm{Add}(\alpha, 1)^{V[G_\alpha]}$. Consider the function $f\colon \alpha \to \alpha$ defined in $V[G_\alpha]$ as follows. For each $\beta < \alpha$ and $p \in \mathrm{Add}(\beta, 1)^{V[G_\alpha]}$ let $\beta(p) < \alpha$ be the least for which there is a condition $q_p \in \mathcal{A} \cap \mathrm{Add}(\beta(p), 1)^{V[G]}$ compatible with p. Set

$$f(\beta) := \sup\nolimits_{p \in \mathrm{Add}(\beta,1)^{V[G_\alpha]}} \beta(p).$$

Let C be the club of closure points of f. Note that for each $\beta < \alpha$ regular,

$$\mathcal{A} \cap \mathrm{Add}(\beta, 1)^{V[G_\alpha]} = \mathcal{A} \cap \mathrm{Add}(\beta, 1)^{V[G_\beta]},$$

and no bounded subsets of α are introduced by the forcing passing from $V[G_\beta]$ to $V[G_\alpha]$. Clearly, if $\beta \in C$ then $\mathcal{A} \cap \mathrm{Add}(\beta, 1)^{V[G_\beta]}$ is a maximal antichain for $\mathrm{Add}(\beta, 1)^{V[G_\beta]}$. Let us prove that for a measure-one set of β's, $\mathcal{A} \cap \mathrm{Add}(\beta, 1)^{V[G_\beta]} \in V[G_\beta]$. Once this is established we will be mostly done.

Subclaim 6.26. $X = \{\nu < \alpha \mid \mathcal{A} \cap \mathrm{Add}(\nu, 1)^{V[G_\nu]} \in V[G_\nu]\} \in U(\alpha, \gamma, t)$ *for all* $\gamma < o^{\vec{U}}(\alpha)$ *and all* γ*-coherent sequence* $t \in [\alpha]^{<\omega}$.

Proof of subclaim. Fix an arbitrary t. Let $\dot{\mathcal{A}}$ and $\dot{X}$ be a $\mathbb{G}_\alpha$-names for $\mathcal{A}$ and the above-displayed set, respectively. Let $p \in G_\alpha$ forcing the above about $\dot{\mathcal{A}}$ and $\dot{X}$. We can moreover assume that $\dot{\mathcal{A}} \subseteq V_\alpha$ - this is possible because $\mathbb{G}_\alpha$ is α-cc and $\mathbb{G}_\alpha \subseteq V_\alpha$. In particular, $j^\alpha_\gamma(\dot{\mathcal{A}}) \cap V_\alpha = \dot{\mathcal{A}} \in M_{U(\alpha,\gamma)}$. Thus,

$\dot{\mathcal{A}}_{G_\alpha} \in M_{U(\alpha,\gamma)}[G_\alpha]$. This is still true in any generic extension of $M_{U(\alpha,\gamma)}[G_\alpha]$ by $j^\alpha_\gamma(\mathbb{P}_\alpha)/G_\alpha$. Therefore, there are $q \in G_\alpha$ ($q \leq p$), $\{\langle t, \dot{T}\rangle\} \in \mathbb{P}(\alpha,\gamma)$ and p_ν such that

$$q \cup \{\langle t, \dot{T}\rangle\} \cup p_\nu \Vdash_{j^\alpha_\gamma(\mathbb{G}_\alpha)} j^\alpha_\gamma(\dot{A}) \cap \mathrm{Add}(\alpha,1)^{V[\dot{G}_\alpha]} \in M_{U(\alpha,\gamma)}[\dot{G}_\alpha].$$

Since $j^\alpha_\gamma(q) = q$ forces the same connection between $j(\dot{\mathcal{A}})$ and $j(\dot{X})$, the above shows that $q \cup \{\langle t, \dot{T}\rangle\} \cup p_\nu$ forces "$\alpha \in j^\alpha_\gamma(\dot{X})$". By definition, this is the same as saying that $\dot{X}_{G_\alpha} \in U(\alpha,\gamma,t)$. □

Since $C \cap X \in \bigcap_{t\in[\kappa]^{<\omega}} U(\alpha,\beta,t)$ it follows that there is $\alpha_0 < \alpha$ such that $b_\alpha \setminus \alpha_0 \subseteq C$. For each $\beta \in b_\alpha \setminus \alpha_0$, $\mathcal{A} \cap \mathrm{Add}(\beta,1)^{V[G_\beta]} \in V[G_\beta]$ is a maximal antichain. Hence, $\mathcal{A} \cap \mathrm{Add}(\beta,1)^{V[G_\beta]}$ must include a restriction of the function $f^*_\alpha \restriction \beta$, as this function is a bounded modification of f^*_β which is $V[G_\beta]$-generic by the induction hypothesis. All in all, $\mathcal{A}$ includes a restriction of f^*_α and we are done. □

The proof of Claim 6.25 completes the inductive verification and establishes the proof of Theorem 6.23. □

Corollary 6.27. *Working in V^*, $\mathbb{P}(\kappa,\omega_1)$ projects onto* $\mathrm{Add}(\kappa,1)^{V^*}$.

7. Further Directions

In this last section we should like to draw a few future directions in which the present work could be applied. Our first proposed direction regards the existence of a minimal *Sacks-like* poset that singularizes a measurable cardinal to uncountable cofinalities. This (if feasible at all) will be analogous to the main poset devised in [KRS13]. Thus, we ask:

Question 7.1. Is there a Prikry-type forcing that changes the cofinality of a measurable cardinal to ω_1 whose generic extension does not have proper intermediate inner models?

It is not far-fetched that a tree-like variation of the Gitik forcing non-normal Magidor/Radin forcing presented here may work in this respect.

There is another question that regards the preparation of Lemma 5.1 (first described in [BG21]). This preparation forces with the lottery sum of $\{\mathrm{Add}(\alpha,1),\{\mathbb{1}\}\}$ for every inaccessible $\alpha < \kappa$ and yields a non-normal κ-complete ultrafilter *concentrating on Cohens* (Clause (2) in Setup 1). Namely, if $C = \langle \kappa_n \mid n < \omega\rangle$ is a Prikry sequence for the Tree Prikry forcing $\mathbb{T}_U$ then $f_C := \bigcup_{n<\omega} f_{\kappa_n} \restriction [\kappa_{n-1},\kappa_n)$ is $\mathrm{Add}(\kappa,1)$-generic over V where f_{κ_n} are $\mathrm{Add}(\kappa_n,1)$-generics over an inner model of V arising from the forcing preparation. A natural inquiry is what can be said about f_C and f_C whenever C and C' are mutually generic $\mathbb{T}_U$-generic sequences.

Question 7.2. Suppose C_1, C_2 partition C into two infinite sets, are f_{C_1}, f_{C_2} mutually generic over V?

Similar techniques to the ones developed in this paper permitt to construct Mitchell increasing sequences with several non-normal ultrafilters, each of which concentrating on Cohens.

Question 7.3. Suppose that $U_0 \triangleleft U_1$ are two non-normal ultrafilters on κ concentrating on Cohens. What is the relation between the corresponding Cohen generic functions?

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