MATH 504: PRELIMINARIES

TOM BENHAMOU
UNIVERSITY OF ILLINOIS AT CHICAGO

1. Definition sets

1.1. The list principle.

$$\{a, b, c, ..., z\},\ \{1, 5, 17\},\ \{\{1, 2\}, \{2, 3\}\}$$

Formally, we can define the "List Principle" by

$$a \in \{a_1, ..., a_n\} \equiv a = a_1 \vee a = a_2 ... \vee a = a_n$$

Let us denote the set of *natural numbers* by: $\mathbb{N} = \{0, 1, 2, ...\}$

The membership relation: $a \in A$ is the statement that the object a is a member of the set A

Remark 1.1. Bounded quantifiers: it will be convenient to use the notion of quantifiers which are bounded in a given set A:

$$\forall x \in A.p(x) \equiv \forall x.x \in A \to p(x)$$

$$\exists x \in A.p(x) \equiv \exists x.x \in A \wedge p(x)$$

We think of these quantifiers as quatifiters which range over a given set.

1.2. The separation principle.

Given a set A an a predicate $p(x)$ where x is a free variable in the set A, we can *separate* from A the elements $a \in A$ which satisfy $p(a)$ into a new set. This separated set is denoted by:

$$\{x \in A \mid p(x)\}$$

This reads as "the set of all x in A such that $p(x)$ holds true". Define $a \in \{x \in A \mid p(x)\} \equiv a \in A \wedge p(a)$

1.3. The replacement principle.

Let A be a set and $f(x)$ some operation/ function on the elements of A. We can *replace* every memeber a of the set A by the outcome of the operation $f(a)$ and collect all the outcomes into a new set. This new collection is denoted by:

$$\{f(x) \mid x \in A\}$$

This reads as "the set of all outcomes $f(x)$ where the parameter x runs in the set A". Define $a \in \{f(x) \mid x \in A\} \equiv \exists x \in A.f(x) = a$

Global variables for famous sets:

(1) $\mathbb{N} = \{0, 1, 2, 3, ...\}$

Date: January 10, 2023.

(2) The set of positive natural numbers is: $\mathbb{N}_+ = \{x \in \mathbb{N} \mid x > 0\} = \{1, 2, 3, 4, ...\}$
(3) The set of integers is: $\mathbb{Z} = \{..., -2, -1, 0, 1, 2, ...\}$
(4) The set of fractions/ rational numbers is: $\mathbb{Q} = \{\frac{m}{n} \mid m, n \in \mathbb{Z} \wedge n \neq 0\}$
(5) The set of real numbers is denoted by $\mathbb{R}$. We will formally define the reals only later in this course. Right now, We will simply describe them as numbers which have a (possibly infinite) decimal representation such as: 15.6755897847566372....... Among the real numbers, one can find $\sqrt{2}, \pi, e$. One of the most important properties of the reals is that the rational numbers are dense inside them (we will prove that):

$$\forall r_1, r_2 \in \mathbb{R}. r_1 < r_2 \ \Rightarrow (\exists q \in \mathbb{Q}. r_1 < q < r_2)$$
$$\mathbb{R}_+ = \{x \in \mathbb{R} \mid x > 0\}.$$

(6) The intervals:
   - $(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$ denotes the *open interval* between a and b.
   - $[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$ the *closed interval.*
   - $[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$. Define similarly $(a, b]$.
   - $(a, \infty) = \{x \in \mathbb{R} \mid a < x\}$ is the *infinite ray.* Similarly define $[a, \infty), (-\infty, a), (-\infty, a]$. Note that $(a, \infty]$ is not defined since ∞ is not a natural number.
(7) $\emptyset$ denoted the empty set, which is characterized by the following property: $\forall x. x \notin \emptyset$. Namely, the empty set is a set with no element. It is sometimes convenient to think of $\emptyset = \{\}$.

1.4. Inclusion and the extensionality principle.

Definition 1.2. Let A, B be any sets. We say that A *is included in* B and denote it by $A \subseteq B$ if

$$\forall x. x \in A \Rightarrow x \in B$$

In other words, if every element of A is an element of B. Using bounded quantifiers we can say that $A \subseteq B$ is the statement $\forall x \in A. x \in B$.

Theorem 1.3. *For every set A, $\emptyset \subseteq A$.*

Definition 1.4. We denote by $A \nsubseteq B$ if $\neg(A \subseteq B)$, namely, if $\exists x \in A. x \notin B$. We denote $A \subsetneq B$ if $A \subseteq B$ and $A \neq B$.

1.5. **Set equality.** The extensionality principle is a basic principle (axiom) in set theory which expresses the fact the a set is determined by its elements.

Definition 1.5. The extesionality principle is the fact that for any two sets A, B:

$$A = B \Leftrightarrow (A \subseteq B) \wedge (B \subseteq A)$$

This means that when we wish to prove set equality $A = B$, we do so by proving a *double inclusion.*

1.6. **Set operations.**

Definition 1.6. Let A, B be sets

(1) The *intersection* of the sets is defined by $A \cap B = \{x \mid x \in A \wedge x \in B\}$.
(2) The *union* of the two sets is denoted by $A \cup B = \{x \mid x \in A \vee x \in B\}$
(3) The *difference* of the sets is defined by $A \setminus B = \{x \in A \mid x \notin B\}$
In the literature, difference of sets is sometimes denoted by $A - B$.
(4) The *complement* of A inside a supset U of A is denoted by $A^c = U \setminus A$. This is conceptually different from difference since we assume that U is some framework set and then A^c is an operation on a single set.
(5) The *symmetric difference* of the sets is denoted by $A \Delta B = (A \setminus B) \cup (B \setminus A)$.

Proposition 1.7. *Sets operations identities:*

(1) *Associativity:*
 (a) $A \cap (B \cap C) = (A \cap B) \cap C$.
 (b) $A \cup (B \cup C) = (A \cup B) \cup C$.
 (c) $A \Delta (B \Delta C) = (A \Delta B) \Delta C$.
(2) *Commutativity:*
 (a) $A \cap B = B \cap A$.
 (b) $A \cup B = B \cup A$.
 (c) $A \Delta B = B \Delta A$.
(3) *Distributivity:*
 (a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
 (b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
(4) *Identities of difference and De-Morgan low's for sets:*
 (a) $A \setminus B = A \cap B^c$.
 (b) $(A \cup B)^c = A^c \cap B^c$.
 (c) $(A \cap B)^c = A^c \cup B^c$.
 (d) $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$
 (e) $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$.
(5) *Identities of the empty set:*
 (a) $A \cap \emptyset = \emptyset$.
 (b) $A \cup \emptyset = A$.
 (c) $A \setminus \emptyset = A$.
 (d) $\emptyset \setminus A = \emptyset$.
 (e) $A \Delta \emptyset = A$.
(6) *Identities of a set and itself:*
 (a) $A \cap A = A$.
 (b) $A \cup A = A$.
 (c) $A \setminus A = \emptyset$.
 (d) $A \Delta A = \emptyset$.

Proposition 1.8. $A = B \Leftrightarrow A \Delta B = \emptyset$

Proposition 1.9. *The following are equivalent:*

(1) $A \subseteq B$

(2) $A \cap B = A$
(3) $A \setminus B = \emptyset$
(4) $A \cup B = B$

1.7. **The power set.**

Definition 1.10. Let A be any set. define the *power set* of A as the set f all possible subsets of A. We denote it by

$$P(A) = \{x \mid x \subseteq A\}$$

Definition 1.11. For a finite set A, we denote be $|A|$ the number of elements in the set A. For example $|\{1, 2, 3, 18, -3\}| = 5$ and $|(-5, 5) \cap \mathbb{Z}| = 9$.

Theorem 1.12. *Let A be a finite set then $|P(A)| = 2^{|A|}$.*

1.8. **Ordered pairs and Cartesian product.**

Definition 1.13. Let x, y be two objects, the *ordered pair* of x and y is defined by $\langle x, y\rangle = \{\{x\}, \{x, y\}\}$.

The basic property of pairs is the following property for which we omit the proof:

Theorem 1.14 (Eauality of pairs)**.** *For every a, b, c, d*

$$\langle a, b\rangle = \langle c, d\rangle \Leftrightarrow a = c \wedge b = d$$

Definition 1.15. Let A, B be two sets. The *Cartesian product* of the sets (named after René Descartes) is defined by $A \times B = \{\langle a, b\rangle \mid a \in A, B \in B\}$

Also define the *square* of a set A is to be $A \times A$.

The *Real plane* is defined to be the set $\mathbb{R}^2$.

Definition 1.16. Let us define by recursion an n-tuple. A 1-tuple is defined by $\langle a\rangle = a$. Given we have defined an n-tuple, we define $n + 1$-tuples using n-tuples and ordered pairs we have already defined.:

$$\langle a_1, ..., a_n, a_{n+1}\rangle = \langle\langle a_1, ...a_n\rangle, a_{n+1}\rangle$$

Example 1.17.
(1) $\langle a_0\rangle = a_0$.
(2) Note that a 2-tuple is the same as an ordered pairs. Indeed, let us denote momentarily the 2-tuple by $\langle a_0, a_1\rangle^*$, then we have

$$\langle a_0, a_1\rangle^* = \langle\langle a_0\rangle, a_1\rangle = \langle a_0, a_1\rangle$$

.

(3) $\langle a_0, a_1, a_2\rangle = \langle\langle a_0, a_1\rangle, a_2\rangle =$

$$\{\{\langle a_0, a_1\rangle\}, \{\langle a_0, a_1\rangle, a_2\}\} = \{\{\{\{a_0\}, \{a_0, a_1\}\}\}, \{\{\{a_0\}, \{a_0, a_1\}\}, a_2\}\}$$

(4) $\langle a_0, a_1, a_2, a_3\rangle = \langle\langle\langle a_0, a_1\rangle, a_2\rangle, a_3\rangle$

Definition 1.18. $\prod_{i=1}^{n} A_i = A_1 \times A_2 \times .... \times A_n = \{\langle \alpha_1, ..., \alpha_n\rangle \mid \alpha_i \in A_i\}$

$A^n = \prod_{i=1}^{n} A$

2. Relations

Definition 2.1. A relation from the set A to the set B is set $R \subseteq A \times B$.

Definition 2.2. Let R be a relation from A to B. Denote:

(1) $aRb \Leftrightarrow \langle a, b\rangle \in R$.
(2) $dom(R) = \{a \in A \mid \exists b \in B.\langle a.b\rangle \in R\}$.
(3) $Im(R) = \{b \in B \mid \exists a \in A.\langle a, b\rangle \in R\}$.
(4) $R^{-1} = \{\langle b, a\rangle \mid \langle a, b\rangle \in R\}$.
(5) $Id_A = \{\langle a, a\rangle \mid a \in A\}$.
(6) If S is a relation from B to C we define:

$$S \circ R = \{\langle a, c\rangle \in A \times C \mid \exists b \in B.\langle a, b\rangle \in R \wedge \langle b, c\rangle \in S\}$$

Proposition 2.3. *(1)* $(R^{-1})^{-1} = R$.
(2) $R \circ Id_A = R$, $Id_B \circ R = R$.
(3) $(T \circ S) \circ R = T \circ (S \circ R)$.
(4) $(S \circ R)^{-1} = R^{-1} \circ S^{-1}$

2.1. **Relations over a single set.**

Definition 2.4. A relation R from A to A (i.e. $R \subseteq A^2$) is called a relation *on the set* A.

Definition 2.5 (Properties of relations and equivalence relation)**.** Let R be a relation on a set A. We say that:

(1) R is *reflexive* (on A) if: $\forall a \in A.aRa$.
(2) R is *symmetric* if: $\forall a, b \in A.aRb \Rightarrow bRa$.
(3) R is *transitive* if: $\forall a, b, c \in A.(aRb) \wedge (bRc) \Rightarrow aRc$.
(4) R is *anti reflexive* if: $\forall x.\langle x, x\rangle \notin R$.
(5) R is *weekly anti symmetric* if $\forall a, b \in A.aRb \wedge bRa \Rightarrow a = b$.
(6) R is *strongly anti symmetric* if $\forall a, b \in A.aRb \Rightarrow bRa$
(7) R is an *equivalence relation* if it is reflexive, symmetric and transitive.
(8) R is an *weak order* if R transitive, reflexive and weakly anti symmetric.
(9) R is *strong order* if R is transitive and strongly anti symmetric.
(10) An order R (either weak or strong) is total/linear if every two elements are comparable, namely:

$$\forall a, b \in A.a = b \vee aRb \vee bRa$$

2.2. **quivalence relations.**

Definition 2.6. Let E be an equivalence relation on a set A. The *equivalence class* of an element $a \in A$ is the set of all conditions $b \in A$ such that a is E-equivalent to b. Formally, we denote the equivalence class of a by

$$[a]_E = \{b \in A \mid aEb\}$$

Proposition 2.7. *Let E be an equivalence relation on A. Then for every $a, b \in A$:*

(1) Either $[a]_E = [b]_E$.
(2) Or $[a]_E \cap [b]_E = \emptyset$

Moreover, $[a]_E = [b]_E$ *if and only if* aEb.

Corollary 2.8. *The following are equivalent:*

(1) $a \not E b$.
(2) $[a]_E \neq [b]_E$.
(3) $[a]_E \cap [b]_E = \emptyset$.

Definition 2.9. Let E be an equivalence relation on A. The *quotient set* of A by E (a.k.a "A modulo E") is the set of **all** equivalence classes.[1]. We denote it by[2]

$$A/E = \{[a]_E \mid a \in A\}$$

Theorem 2.10. *A/E is a partition of A and any partition of A is induced from some equivalence relation.*

2.3. **orders.**

2.4. **Functions.**

Definition 2.11. Let A, B be two sets. A relation R from A to B is called:

(1) *Total* on A, if $\forall a \in A.\exists b \in B.aRb$.
(2) *univalent*, if $\forall a \in A.\forall b_1, b_2 \in B.aRb_1 \wedge aRb_2 \Rightarrow b_1 = b_2$.
(3) A *function* from A to B if it is total and univalent.

Notation 2.12. If f is a function from A to B we denote it by $f : A \to B$. Also if $f : A \to B$ is a function, we denote $f(a) = b$ if and only if $\langle a, b\rangle \in f$. So $f(a)$ is the unique object in the set B that the function f attaches to the element a.

Definition 2.13. A sequence of elements in the set A is a function $f : \mathbb{N} \to A$. In calculus we sometime denote $a_n = f(n)$ and $(a_n)_{n=0}^{\infty} = f$.

Remark 2.14. Let $f : A \to B$ be a function. The *domain* of f is simply A, we denote $dom(f) = A$. The *range* of f is B and we denote $\mathrm{Im}(f) = B$. The *image* of f is the set $img(f) = \{f(a) \mid a \in A\}$.

Definition 2.15. Let A, B be two sets. We denote *the set of all functions* from A to B by

$${}^AB = \{f \in P(A \times B) \mid f \text{ is a function from } A \text{ to } B\}$$

Theorem 2.16 (Functions equality)**.** *Let f, g be two function. Then the following are equivalent:*

(1) $\mathrm{dom}(f) = \mathrm{dom}(g)$ *and* $\forall x \in \mathrm{dom}(f).f(x) = g(x)$.
(2) $f = g$.

Here are some of the most common ways to define functions in this wat:

---

[1]Needless to say, without repetitions.

[2]Do not confused A/E with set difference $A \setminus E$.

(1) Defining a function with a formula: The definition has the form " Define $f : A \to B$ by $f(a) =$(some formula)". For example, we can define $f : \mathbb{R} \to \mathbb{R}$ by $f(r) = 2$, this is the constant function which for every real r returns the value 2. Another example, define $g : P(\mathbb{N}) \to P(\mathbb{N})$ by $g(X) = X \cup \{1, 2\}$. Then for example $g(\{1, 3, 4\}) = \{1, 2, 3, 4\}$ and $g(\mathbb{N}) = \mathbb{N}$.

Important: If we define $f : A \to B$ by a formula $f(a) =$(some formula) we **must** always make sure that the functions we define are well defined in the sense that:

  (a) The function is total. Practically, this means that we should make sure that the formula for $f(a)$ is defined for every $a \in A$.
  (b) The function is univalent. This means that for every $a \in A$, the formula for $f(a)$ points to a single element. (This is trivial in most cases)
  (c) for every $a \in A$ the formula for $f(a)$ returns an element in B. So the ranged we declared when we wrote $f : A \to B$ is indeed correct.

(2) Definition of a function by cases: Suppose we which to define a function on a set A, and for some of the elements of A we want one formula and for the another part of A we want to use a different formula. We can do that the following way: "Define $f : A \to B$ by

$$f(a) = \begin{cases} \text{(first formula)} & \text{(first condition on } a\text{)} \\ \text{(second formula)} & \text{(second condition on a)} \\ \ldots \end{cases}$$

Definition 2.17. Let $f : A \to B$ be a function and $X \subseteq A$. We define the *restriction of f to X*, denote by $f \restriction X : X \to B$, and a function with domain $dom(f \restriction X) = X$ and for every $x \in X$, $(f \restriction X)(x) = f(x)$.

Definition 2.18. Let $f : A \to B$ be a function we sat that f is:

(1) One to one/ injective: if for every $a_1, a_2 \in A$, if $f(a_1) = f(a_2)$ then $a_1 = a_2$.
(2) Onto/ surjective: if for every $b \in B$ there is $a \in A$ such that $f(a) = b$.

Theorem 2.19. *Let $f : A \to B$ and $g : B \to C$ be two functions. Then the composition of g in f is a function $g \circ f : A \to C$, with domain A and range C such that for each $a \in A$, $(g \circ f)(a) = g(f(a))$.*

Moreover, the composition of $1-1$/ onto is $1-1$/onto

Definition 2.20. A function $f : A \to B$ is invertible if there is a function $g : B \to A$ such that:

$$g \circ f = id_A \quad \text{and } f \circ g = id_B$$

Theorem 2.21. *If g_1, g_2 are two inverse functions of f then $g_1 = g_2$. Moreover, the inverse function of f is the relation f^{-1}.*

Theorem 2.22. *A function $f : A \to B$ is invertible if and only if it is one-to-one and onto.*

3. Equinumerability

Definition 3.1. Let A, B be any sets. We say that:

(1) $|A| = |B|$ "A, B have the same cardinality" if there is $f : A \to B$ which is invertible.
(2) $|A| \leq |B|$ "the cardinality of A is at most the cardinality of B" if there is $f : A \to B$ which is injective.
(3) $|A| \neq |B|$ if $\neg(|A| = |B|)$, namely if there is no invertible $f : A \to B$.
(4) $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$.

Claim 3.1.1. *for any sets A, B:*

(1) $A \subseteq B \to |A| \leq |B|$.
(2) $|A| = |A|$.
(3) $|A| = |B| \to |B| = |A|$.
(4) $|A| = |B| \wedge |B| = |C| \to |A| = |C|$.
(5) $|A| \leq |B| \leq |C| \to |A| \leq |C|$.
(6) $|A| = |B| < |C| \to |A| < |C|$.
(7) $|A| < |B| = |C| \to |A| < |C|$

Claim 3.1.2. (AC) $|A| \leq |B|$ *iff there is* $f : B \to A$ *onto.*

Proposition 3.2. *Let A, A', B, B' be sets such that $|A| = |A'|$ and $|B| = |B'|$. Then:*

(1) $|P(A)| = |P(A')|$.
(2) $|A \times B| = |A' \times B'|$.
(3) $|{}^{B}A| = |{}^{B'}A'|$.
(4) If A, B are disjoint and A', B' are disjoint then $|A \uplus B| = |A' \uplus B'|$.

Theorem 3.3 (Cantor-Berstein). *Let A, B be sets and supose that $|A| \leq |B| \wedge |B| \leq |A|$ then $|A| = |B|$.*

Corollary 3.4. *If $|A| < |B| \leq |C|$ or $|A| \leq |B| < |C|$ then $|A| < |C|$.*

Theorem 3.5 (Cantor-Schröeder-Bernstein). *If $|A| \leq |B|$ and $|B| \leq |A|$ then $|A| = |B|$.*

Definition 3.6. A set A is countable if $|A| = |\mathbb{N}|$ and we denote it by $|A| = \aleph_0$.

Theorem 3.7. *(AC) If A is infinite then $\aleph_0 \leq |A|$.*

Theorem 3.8. *The following sets are countable:* $\mathbb{Z}, \mathbb{N}_{even}, \mathbb{Q}, \mathbb{N}\times\mathbb{N}, \mathbb{N}^n, \{X \in P(\mathbb{N}) \mid X \textit{ is finite }\}$

Theorem 3.9. *The countable union of countable sets if countable*

Theorem 3.10 (Cantor's Diagonalization Theorem)**.** $\aleph_0 < |{}^{\mathbb{N}}\{0,1\}|$

Definition 3.11. $2^{|A|} = |{}^A\{0,1\}|$

Theorem 3.12. $|P(A)| = 2^{|A|}$

Theorem 3.13 (Cantor's Theorem)**.** $|A| < 2^{|A|}$

Theorem 3.14. $|\mathbb{R}| = 2^{\aleph_0}$, $|\mathbb{R}^n| = 2^{\aleph_0}$.

Theorem 3.15. $|[\alpha, \beta]| = |(\alpha, \beta)| = |(\alpha, \infty)|$