PROBLEM SET 1

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Assume $Ax0, Ax_1, Ax3 - Ax6$.

Problem 1. Prove that for any set A, the identity function $Id_A = \{\langle a, a\rangle \mid a \in A\}$ exists and is unique.

Problem 2. Prove that for any function $f : A \to B$ then for every $Y \in B$, $f^{-1}[Y]$ exists.

Problem 3. Prove that if E is an equivalence relation over a set A then $A/E := \{[a]_E \mid a \in A\}$ exists.

Problem 4. Let $\langle A, R\rangle$ be a well-ordered set and assume that A is infinite. Prove that either $\langle A, R\rangle \simeq \langle \mathbb{N}, < \rangle$ or there is $a \in A$ such that $\langle A_R[a], R\rangle \simeq \langle \mathbb{N}, < \rangle$. [Hint: The tricheotomy theorem]

Problem 5. On $\mathbb{N}\times\mathbb{N}$ define the order $\langle n, m\rangle \prec \langle n', m'\rangle$ iff $\max(n, m) < \max(n', m')$ or $\max(n, m) = \max(n', m')$ and $\langle n, m\rangle <_{Lex} \langle n', m'\rangle$. Prove that $\prec$ is a well order of $\mathbb{N} \times \mathbb{N}$ and that $\langle \mathbb{N} \times \mathbb{N}, \prec \rangle \simeq \langle \mathbb{N}, < \rangle$.

Problem 6. Prove that if $\langle A, R\rangle$ is a well-ordered set and $f : A \to A$ is an order isomorphism then $f = Id$. Can you find a counterexample if A is not well-ordered?

Problem 7. Prove that $Ax0, Ax1, Ax3, Ax4$ are not enough to prove that there is a set with three elements.

[Hint: Let $T_0 = \emptyset$ and recursively define $T_{n+1} = \{X \subseteq T_n \mid |X| \leq 2\}$ and $T = \cup_{n<\omega} T_n$. Consider the model $\mathcal{M} = (T, \in)$.]