MidTerm I- Mathematical Logic-Solutions

MATH 461 (Instructor: Tom Benhamou) Feb 26

Problems

Problem 1. Prove that $\langle \mathbb{N}\times\mathbb{N}, <_{LEX}\rangle \not\simeq \langle \mathbb{Z}, <\rangle$. (28 pt.)

[Recall: $<_{LEX}$ denotes the lexicographic order on $\mathbb{N}\times\mathbb{N}$ and $<$ is the regular order on the integers.]

Solution. Suppose towards a contradiction that the two orders are isomorphic and let $f : \mathbb{N}\times\mathbb{N} \longrightarrow \mathbb{Z}$ be an isomorphism. Consider $f(\langle 0,0\rangle) = z \in \mathbb{Z}$, then $z-1 \in \mathbb{Z}$ and since f is an isomorphism, there is $\langle n, m \in \mathbb{N}\times\mathbb{N}$ such that $f(\langle n,m\rangle) = z-1$. since f is order preserving, $\langle n,m\rangle <_{LEX} \langle 0,0\rangle$. By definition of $<_{LEX}$, either $n<0$ or $n=0 \wedge m<0$ in either case we obtain a contradiction, as both n,m are natural numbers.

Problem 2. On $P(\mathbb{N})$, consider the relation

$$E = \Big\{\langle A,B\rangle \in P(\mathbb{N})^2 \mid (A\Delta B)\cap\{2023,2024\} = \emptyset\Big\}.$$

[Recall: the symmetric difference is defined by $A\Delta B = (A\setminus B)\cup(B\setminus A)$.]

(a) Prove that E is an equivalence relation on $P(\mathbb{N})$. (15 pt.)

(b) How many elements are there in the set $P(\mathbb{N})/E$? Provide a system of representatives for E. No proof required. (20 pt.)

[Instructions: your answer should look like "$P(\mathbb{N})/E$ has ... elements, and a system, of representatives is given by ...."]

Solution. (a) The relation is reflective since for every $A \in P(\mathbb{N})$, $A\Delta A = \emptyset$, and therefore $A\Delta A \cap \{2023,2024\} = \emptyset$. Hence $\langle A,A\rangle \in E$. The relation is symmetric, since if $\langle A,B\rangle \in E$, then $A\Delta B \cap \{2023,2024\} = \emptyset$. Now

$B\Delta A = A\Delta B$ and therefore $B\Delta A \cap \{2023, 2024\} = \emptyset$. Hence $\langle B, A\rangle \in E$. Finally to see that E is transitive, suppose that $\langle A, B\rangle, \langle B, C\rangle \in E$, then $A\Delta B \cap \{2023, 2024\} = \emptyset$ and $B\Delta C \cap \{2023, 2024\} = \emptyset$. Note that $A\Delta C \subseteq A\Delta B \cup B\Delta C$ (prove that!) and therefore

$$A\Delta C \cap \{2023, 2024\} \subseteq (A\Delta B \cup B\Delta C) \cap \{2023, 2024\} =$$

$$= A\Delta B \cap \{2023, 2024\} \cup B\Delta C \cap \{2023, 2024\} = \emptyset \cup \emptyset = \emptyset.$$

(b) $P(\mathbb{N})/E$ has 4 elements. A system of representatives if given by $P(\{2023, 2024\})$.

Problem 3. (a) Formulate Cantor-Berstein Theorem. No proof required. (3 pt.)

(b) A function $f : \mathbb{N} \to \mathbb{N}$ is called *double-valued* if for every $n \in \mathbb{N}$, $|\{m \in \mathbb{N} \mid f(m) = n\}| = 2$. Give an example of a double-valued function. No proof required. (10 pt.)

[Instructions: your solution should look like "Here is my example: define $f : \mathbb{N} \to \mathbb{N}$ by $f(n) = ...$".]

(c) Compute the cardinality of the set of all double-valued functions $f : \mathbb{N} \to \mathbb{N}$. Prove your answer. (30 pt.)

Solution.

(a) clear.

(b) Here is my example: define $f : \mathbb{N} \to \mathbb{N}$ by

$$f(n) = \begin{cases} \frac{n}{2} & n \in \mathbb{N}_{even} \\ \frac{n-1}{2} & n \in \mathbb{N}_{odd} \end{cases}$$

(c) Let A denote the set of all double-valued functions. Let us prove that the cardinality of A is $2^{\aleph_0}$ using Cantor-Bernstein. To see this, first note that $A \subseteq {}^{\mathbb{N}}\mathbb{N}$ and therefore $|A| \leq |{}^{\mathbb{N}}\mathbb{N}|$. We saw in class that $|{}^{\mathbb{N}}\mathbb{N}| = 2^{\aleph_0}$. Now for the other direction, let us define a function $F : {}^{\mathbb{N}}\{0,1\} \to A$.

The idea is to code each value $f(n)$ by changing which elements are mapped to $2n$ and which are mapped to $2n+1$.

For each $f : \mathbb{N} \to \{0,1\}$, we define $F(f)(4n) = F(f)(4n+1) = 2n$ and $F(f)(4n+2) = F(f)(4n+3) = 2n+1$ if $f(n) = 0$ and $F(f)(4n) = F(f)(4n+3) = 2n+1$ and $F(f)(4n+1) = F(f)(4n+2) = 2n$ if $f(n) = 1$. A formula which does it is as follows:

$$F(f)(m) = \begin{cases} \frac{m-1}{2} & m \in \mathbb{N}_{odd} \\ \frac{m+1-(-1)^{f(\frac{m}{4})}}{2} & 4|m \\ \frac{m-1+(-1)^{f(\frac{m-2}{4})}}{2} & otherwise \end{cases}$$

It follows that for every n, there are precisely two m's (by splitting into case whether n is even or odd and if $f(\frac{n}{2})$ is 0 or 1) such that $F(f)(m) = n$, and therefore $F(f)$ is double-valued. Also, if $f \neq g$ then for some n, $f(n) \neq g(n)$. WLOG $f(n) = 0$ and $g(n) = 1$. Then $F(f)(4n) = 2n$ and $F(g)(4n) = 2n+1$, hence $F(f)(4n) \neq F(g)(4n)$ and therefore $F(f) \neq F(g)$. It follows that F is injective. It follows that $|{}^{\mathbb{N}}\{0,1\}| \leq |A|$. By Cantor-Bernstein, we conclude that $|A| = 2^{\aleph_0}$.