MidTerm example- Mathematical Logic-Sols

MATH 361 (Instructor: Tom Benhamou) Nov 17

Instructions

The midterm duration is 1 hour and 20 min, and consists of 4 problems, each worth 26 points (The maximal grade is 100). The answers to the problems should be written in the designated areas.

Problems

Problem 1. Let us define recursively $A_0 = \emptyset$ and $A_{n+1} = P(A_n)$. Prove by induction that for every n, $A_n \subseteq A_{n+1}$

Solution: By induction. For $n = 0$ $A_0 = \emptyset$ is a subset of every set and therefore $A_0 \subseteq A_1$. Suppose this is true for $n-1$ and let us prove that $A_n \subseteq A_{n+1}$. Let $X \in A_n = P(A_{n-1})$. Then $X \subseteq A_{n-1}$. By the induction hypothesis, $A_{n-1} \subseteq A_n$ and therefore $X \subseteq A_n$. By definition $X \in P(A_n) = A_{n+1}$. It follows that $A_n \subseteq A_{n+1}$.

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Problem 2. Give the definition of a linear ordering.

(a) Give the definition of an isomorphism between two linear orders $\langle L_1, <\rangle$ and $\langle L_2, \prec\rangle$.

Prove or disprove each of the following statements:

(i) $\langle \mathbb{Q} \setminus \mathbb{Z}, <\rangle \simeq \langle \mathbb{Q} \setminus \mathbb{N}, <\rangle$.

Solution. Use Cantor's theorem to prove they are isomorphic.

(ii) $\langle \mathbb{R}, <\rangle \simeq \langle \mathbb{R} \setminus (0, 1), <\rangle$.

Solution. Not isomorphic. Suppose otherwise, then there is an isomorphism $f : \mathbb{R} \setminus (0, 1) \to \mathbb{R}$. Consider $f(1) = r$ and $f(0) = r'$. Since $0 < 1$, and f is order preserving, $r' < r$. Let $r' < r'' < r$ be any real, then since f is subjective, there is $q \in \mathbb{R} \setminus (0, 1)$ such that $f(q) = r''$, but since $f(0) < f(q) < f(1)$, $0 < q < 1$, contradiction.

Solution:

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Problem 3. Fix a natural number $N > 0$. A function $f : \mathbb{N} \to \{0,1\}$ is called N-periodic if for every $n \in \mathbb{N}$, $f(n+N) = f(n)$. For any $N > 0$, let A_N be the set of all N-periodic functions. Show that

$$A_N \approx \{0,1\}^N = \{0,1\} \times \{0,1\} \times ... \times \{0,1\}$$

[Instructions: Half of the points are given for a correct definition of a bijection, the other half is the proof that the defined function is indeed a bijection.]

Solution

The function $F : A_N \to \{0,1\}^N$ defined by $F(f) = \langle f(0), ..., f(N-1)\rangle$ is one-to-one and onto. to see this, let $f_1, f_2 \in A_N$ and suppose that $F(f_1) = F(f_2)$. Then for every $0 \leq i < N$, $f_1(i) = f_2(i)$. For $n \geq N$, since f_1, f_2 are N-periodic, $f_1(n) = f_1(n \mod N) = f_2(n \mod N) = f_2(n)$. To see that F is onto, let $\langle a_0, ..., a_{N-1}\rangle \in \{0,1\}^N$. Define $f \in A_N$ by $f(n) = a_{n \mod N}$. Then f is N-periodic and $F(f) = \langle a_0, .., a_{N-1}\rangle$. Hence F is onto.

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Problem 4. A function f is called periodic if there is $N \in \mathbb{N}_+$ such that f is N-periodic. Show that the set A of all N-periodic functions is infinitely countable. [Remark: You can use Problem 3 even if you did not prove it.]

Solution First we note that the function $F : \mathbb{N}_+ \to A$ defined by $F(N)(m) = \begin{cases} 1 & n \mod N = 0 \\ 0 & o.w. \end{cases}$. (namely, $F(N)$ is the indicator function for the set of n which are divisible by N) is a well-defined function. To see this, we claim that $F(N)$ is N-periodic. Indeed, for every n, $F(n) = 1$ is and only if n is divisible by N if and only if $n + N$ is divisible by N if and only if $F(N)(n + N) = 1$. Also, it is one-to-one since if $n \neq m$ then without loss of generality, $n < m$ and therefore $F(n)(n) = 1$ while $f(m)(n) = 0$. So $F(n) \neq F(m)$. We conclude that $\mathbb{N} \approx \mathbb{N}_+ \preceq A$. For the other direction, $A = \bigcup_{N \in \mathbb{N}_+} A_N$, and by the previous problem each A_N is a finite set and in particular countable. We conclude that A is a countable union of countable sets hence countable. By CSB A is infinitely countable.