# Homework 8

MATH 461 (due April 5) March 29, 2024

**Problem 1.** Let $C$ be an axiomatizable class of structures for the first-order language $\mathcal{L}$ and let $\mathfrak{a}, \mathfrak{b}$, be any structures for the language $\mathcal{L}$. Prove that if $\mathfrak{a} \equiv \mathfrak{b}$, then $\mathfrak{a} \in C$ if and only if $\mathfrak{b} \in C$

**Problem 2.** Let $\mathcal{L}$ be a first-order language and let $C$ be any class of $\mathcal{L}$-structures. Show that $C$ is finitely axiomatizable if and only if $C$ is axiomatizable by a single formula.

**Problem 3.** Let $\mathcal{L}$ be a first-order language and let $C$ be an axiomatizable class of $\mathcal{L}$-structures. Suppose that $C' \subseteq C$ is finitely axiomatizable, and prove that $C \setminus C'$ is axiomatizable.

**Problem 4.** Let $F$ be a field. Consider the language of $F$-vector spaces $\mathcal{L}_{VS}^{F} = \{c_0, +\} \cup \{f_r \mid r \in F\}$. Where $c_0$ (intended to be the 0-vector) is a constant symbol, $+$ is a 2-placed function symbol (intended to be vector addition) and $f_r$ is a 1-places function symbol (intended to be the scalar multiplication of a vector by $r$).

(1) Explain (Namely, describe the interpretation of each non-logical symbol of the language) how the usual $n$-real-tuples vector space (i.e. $\mathbb{R}^n$) is an $\mathcal{L}_{VS}^{\mathbb{R}}$-structure.

(2) Explain how the usual set of finite degree polynomials with real coefficients (i.e. $\mathbb{R}[X]$) is an $\mathcal{L}_{VS}^{\mathbb{R}}$-structure.

(3) Prove that the class $C$ of real-valued vector spaces is axiomatizable.

[For your convenience: vector spaces-axioms]

(4) Let $F$ be a <u>finite</u> field. Prove that the class of infinite dimensional vector spaces over $F$ is axiomatizable.

[Recall: An infinite dimensional vector space is a vector space with no finite base. Equivalently, if for every $n \in \mathbb{N}$ there is a linearly independent set containing $n$-many vectors.]

[Hint: Formulate the statement $\Theta_n$ which states that there are $n$-many linearly independent vectors.]

(5) Prove that the class of finite dimensional vector spaces over $F$ is not axiomatizable and deduce that the class of infinite dimensional vector spaces is not finitely axiomatizable.