Homework 8

MATH 461 (due April 5) March 29, 2024

Problem 1. Let C be an axiomatizable class of structures for the first-order language $\mathcal{L}$ and let $\mathfrak{a}, \mathfrak{b}$, be any structures for the language $\mathcal{L}$. Prove that if $\mathfrak{a} \equiv \mathfrak{b}$, then $\mathfrak{a} \in C$ if and only if $\mathfrak{b} \in C$

Solution. Let T be a theory such that $C = Mod(T)$. Let $\mathfrak{a}, \mathfrak{b}$ be $\mathcal{L}$-structure such that $\mathfrak{a} \equiv \mathfrak{b}$. Then $\mathfrak{a} \in C$ iff (by definition of $Mod(T)$) $\mathfrak{a} \models T$ iff (by definition of $\mod T$) for all $\sigma \in T$ $\mathfrak{a} \models \sigma$ iff (by elementary equivalence) for all $\sigma \in T$ $\mathfrak{b} \models \sigma$ iff $\mathfrak{b} \models T$ iff $\mathfrak{b} \in C$, as wanted.

Problem 2. Let $\mathcal{L}$ be a first-order language and let C be any class of $\mathcal{L}$-structures. Show that C is finitely axiomatizable if and only if C is axiomatizable by a single formula.

Solution. If it is axiomatizable by a single sentence, then it is finitely axiomatizable. If C is finitely axiomatizable by $\{\sigma_1, .., \sigma_n\}$, consider the sentence $\sigma = \sigma_1 \wedge ... \wedge \sigma_n$. Then for every $\mathcal{L}$-structure $\mathfrak{a}$, by definition of $\wedge$, $\mathfrak{a} \models \sigma$ iff for every $1 \leq i \leq n$, $\mathfrak{a} \models \sigma_i$ iff $\mathfrak{a} \models \{\sigma_1, ...\sigma_n\}$ iff $\mathfrak{a} \in C$. Hence $\{\sigma\}$ is an axiomatization of C.

Problem 3. Let $\mathcal{L}$ be a first-order language and let C be an axiomatizable class of $\mathcal{L}$-structures. Suppose that $C' \subseteq C$ is finitely axiomatizable, and prove that $C \setminus C'$ is axiomatizable.

Solution Let T be an axiomatization for C and suppose that $\{\sigma_1, ..., \sigma_n\}$ is a finite axiomatization for $C' \subseteq C$. Let $\phi = \neg\sigma_1 \vee \neg\sigma_2 \vee ... \vee \neg\sigma_n$ and consider $T' = T \cup \{\phi\}$. It is not hard to check that T' is an axiomatization of $C \setminus C'$.

Problem 4. Let F be a field. Consider the language of F-vector spaces $\mathcal{L}_{VS}^{F} = \{c_0, +\} \cup \{f_r \mid r \in F\}$. Where c_0 (intended to be the 0-vector) is a constant symbol, + is a 2-placed function symbol (intended to be vector addition) and f_r is a 1-places function symbol (intended to be the scalar multiplication of a vector by r).

(1) Explain (Namely, describe the interpretation of each non-logical symbol of the language) how the usual n-real-tuples vector space (i.e. $\mathbb{R}^n$) is an $\mathcal{L}_{VS}^{\mathbb{R}}$-structure.

Solution. The universe of $\mathfrak{a}$ is $\mathbb{R}^n$, $c_0^{\mathfrak{a}} = \vec{0}$ is the 0-vector. $+^{\mathfrak{a}}$ is usual n-tuples addition (i.e. coordinatewise) and $f_r^{\mathfrak{a}} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is defined by $f_r^{\mathfrak{a}}(\vec{v}) = r \cdot \vec{v}$, where $\cdot$ is the usual scalar multipliction.

(2) Explain how the usual set of finite degree polynomials with real coefficients (i.e. $\mathbb{R}[X]$) is an $\mathcal{L}_{VS}^{\mathbb{R}}$-structure.

Solution. Similar to the previous ite.

(3) Prove that the class C of real-valued vector spaces is axiomatizable.

[For your convenience: vector spaces-axioms]

Solution. The axioms described in the reference is an axiomatization of vector spaces.

(4) Let F be a finite field. Prove that the class of infinite dimensional vector spaces over F is axiomatizable.

[Recall: An infinite dimensional vector space is a vector space with no finite base. Equivalently, if for every $n \in \mathbb{N}$ there is a linearly independent set containing n-many vectors.]

[Hint: Formulate the statement Θ_n which states that there are n-many linearly independent vectors.]

Solution. Let

$$\Theta_n = \exists x_1 \exists x_2 \ldots \exists x_n \bigwedge_{\langle a_1,\ldots,a_n\rangle \in F^n \setminus \{\vec{0}\}} f_{a_1}(x_1) + \ldots + f_{a_n}(x_n) \neq 0.$$

Note that Θ_n is indeed a (finite) WFF since F is a finite set. Then by the hint $\{\Theta_n \mid n \in \mathbb{N}\}$ together with the axiomatization of F-vector space is an axiomatization of infinite dimensional F-vector spaces.

(5) Prove that the class of finite dimensional vector spaces over F is not axiomatizable and deduce that the class of infinite dimensional vector spaces is not finitely axiomatizable.

Solution. Suppose toward contradiction that the class of finite dimensional F-vector spaces is axiomatizable by T, Then $T \cup \{\Theta_n \mid n < \omega\}$ is finitely satisfiable (the models F^n witness that). By the compactness theorem, $T \cup \{\Theta_n \mid n < \omega\}$ has a model V, then V is supposed to have finite dimension (as it satisfies T) but also it satisfies Θ_n for all n so it had infinite dimension, contradiction. We conclude by the previous problem that the infinite dimensional vector spaces are not finitely axiomatizable.