Homework 7

MATH 461 (due March 29) March 22, 2024

Problem 1. Let $\mathcal{L}$ be the language where the non-logical 2-places function symbols are $+, \times$ and $\bar{1}$ is a non-logical constant symbol and there are no predicate symbols.

(a) Let $\mathfrak{a} = \langle \mathbb{R}, +^{\mathfrak{a}}, \times^{\mathfrak{a}}, \bar{1}^{\mathfrak{a}} \rangle$ be the $\mathcal{L}$-structure where $+^{\mathfrak{a}}, \times^{\mathfrak{a}}, \bar{1}^{\mathfrak{a}}$ are the usual $+, \cdot, 1$ on reals. Denote by $\bar{n}$ the term $\underbrace{\bar{1} + \bar{1} + ... + \bar{1}}_{n\text{-times}}$. Find a term for $2x^2 + x + 1$. Describe (without proof) all the terms.

Solution Here is a term for $2x^2 + x + 1$ $+(+(\times(\bar{2}, \times(x, x)), x), 1)$.

A general term is a polynomial with a positive natural number coeficient.

(b) Find a WFF which expresses that $\sqrt{2}$ exists.

Solution. $\exists x(x \times x = \bar{2})$

Problem 2. Let $\mathfrak{a} = \langle A^{\mathfrak{a}}, ... \rangle$ be a $\mathcal{L}$-structure and let t be a term. If $s_1, s_2 \colon V \longrightarrow A^{\mathfrak{a}}$ agree on all variables (if any) in t, then $\bar{s}_1(t) = \bar{s}_2(t)$. (Hint: argue by induction on the length of t.)

Solution. For $t = x$ a variable, $s_1(x) = s_2(x)$ by assumption, and therefore, $\bar{s}_1(x) = s_1(x) = s_2(x) = \bar{s}_2(x)$. If $t = c$ is a constant symbol, then $\bar{s}_1(c) = c^{\mathfrak{a}} = \bar{s}_2(c)$. If $t = f(t_1, ..., t_n)$ where f is a n-placed function symbol and $t_1, ..., t_n$ are terms, then by the induction hypothesis:

$$\bar{s}_1(t) = f^{\mathfrak{a}}(\bar{s}_1(t_1), ..., \bar{s}_1(t_n)) = f^{\mathfrak{a}}(\bar{s}_2(t_1), ..., \bar{s}_2(t_n)) = \bar{s}_2(t)$$

Definition. Suppose that $\mathfrak{a}, \mathfrak{b}$ are structures for the first order language $\mathcal{L}$. Then $\mathfrak{a}$ and $\mathfrak{b}$ are said to be elementarily equivalent, written $\mathfrak{a} \equiv \mathfrak{b}$, if for every sentence σ,

$$A \models \sigma \Leftrightarrow B \models \sigma.$$

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Problem 3. Let $\mathcal{L}$ be the first order language such that the only nonlogical symbol is the 2-place predicate symbol $<$. Let $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ be the following $\mathcal{L}$-structures:

- $\mathfrak{a} = \langle \mathbb{N}, <^{\mathfrak{a}} \rangle$.
- $\mathfrak{b} = \langle \mathbb{Z}, <^{\mathfrak{b}} \rangle$.
- $\mathfrak{c} = \langle \mathbb{Q}, <^{\mathfrak{c}} \rangle$.

where $<^{\mathfrak{a}}, <^{\mathfrak{b}}, <^{\mathfrak{c}}$ are the usual linear orderings of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ respectively. Prove that:

(i) $\mathfrak{a} \not\equiv \mathfrak{b}$

Solution. The sentence $\exists x \forall y (x = y \vee x < y)$ holds in $\mathfrak{a}$ but not in $\mathfrak{b}$

(ii) $\mathfrak{b} \not\equiv \mathfrak{c}$. **solution.** Density.