Homework 6

MATH 461 (due March 22) March 13, 2024

Problem 1. If $\langle T, \prec \rangle$ is a tree, then the following are equivalent:

(i) T is finitely branching.

(ii) $\mathcal{L}_n(T)$ is finite for all $n \geq 0$.

Problem 2. Use König's Lemma to prove that if $\langle A, \prec \rangle$ is a countable partial order, then there exists a linear ordering $<$ of A which extends $\prec$.

Problem 3. In class we proved Ramsey's theorem for countable graphs, that is: If $c : [\mathbb{N}]^2 \to \{0, 1\}$ is any coloring, then there is $H \subseteq \mathbb{N}$ infinite such that $c \upharpoonright [H]^2$ is constant. We called such H a c-monochromatic set.

(i) Prove that for any $r \in \mathbb{N}$ and for every $c : [\mathbb{N}]^2 \to \{0, 1, ..., r\}$ there is an infinite set $H \subseteq \mathbb{N}$ which is c-monochromatic.

(ii) (Optional) Prove that for any $r, s \in \mathbb{N}$ and for every $c : [\mathbb{N}]^s \to \{0, 1, ..., r\}$ there is an infinite set $H \subseteq \mathbb{N}$ which is c-monochromatic.

Problem 4. In this exercise, you will prove the Compactness Theorem from König's Lemma. Given a countable $\Sigma \subseteq \overline{\mathcal{L}}$ which is finitely satisfiable, let us define a tree. First we enumerate $\Sigma = \{\sigma_0, \sigma_1, \sigma_2, ...\}$, and let $\mathcal{L} = \{v_0, v_1, ...\}$ be an enumeration of all the sentence symbols.

(a) Each function $\phi : \{0, ..., n\} \to \{0, 1\}$, can be identified with a functions $\phi^* : \Gamma_n \to \{T, F\}$ where $\Gamma_n = \{v_0, ..., v_n\}$. Describe the identification. No proof required.

(b) Let $\Sigma_n \subseteq \Sigma$ be the set of all $\sigma \in \Sigma$ which mentions only sentence symbols from Γ_n. Show by induction on the length of σ, that for any TVA V, the value of $\bar{V}(\sigma)$ depends only on $V \upharpoonright \Gamma_n$, namely if V_1, V_2 are TVA's such that $V_1 \upharpoonright \Gamma_n = V_2 \upharpoonright \Gamma_n$ then $V_1(\sigma) = V_2(\sigma)$.

(c) Define the tree $T \subseteq T_2$ as follows: at level n, we put all the function ϕ such that some (any) TVA which extends ϕ^* satisfies Σ_n. We order T as usual by end-extension of functions. Prove that if T has an infinite branch then Σ is satisfiable.

(d) Show that for every n, $\mathcal{L}_n(T) \neq \emptyset$. Namely, prove that for each n, there is $\phi : \{0, ..., n\} \to \{0, 1\}$ such that ϕ^* extends to a TVA which satisfy Σ_n. This proof is done in a few steps:

  (a) Prove the existence of ϕ^* in case Σ_n is finite.

  (b) If Σ_n is infinite, enumerate $\Sigma_n = \{\sigma_0, \sigma_1, ...\}$ and for each k prove that there is a TVA ϕ_k such that satisfying $\{\sigma_0, ..., \sigma_k\}$.

  (c) Use the pigeonhole principle to find a single $\phi : \{0, ..., n\} \to \{0, 1\}$ such that for infinitely many k's $\phi_k \restriction \{v_0, .., v_n\} = \phi^*$.

  (d) Prove that $\phi \in \mathcal{L}_n(T)$.

(e) Use König's Theorem to show that T has an infinite branch and deduce that Σ is satisfiable.