Homework 6

MATH 461 (due March 22) March 13, 2024

Problem 1. If $\langle T, \prec\rangle$ is a tree, then the following are equivalent:

(i) T is finitely branching.

(ii) $\mathcal{L}_n(T)$ is finite for all $n \geq 0$.

Solution.

(i) Suppose that T is finitely branching, let us prove by induction that $\mathcal{L}_n(T)$ is finite for every n. For $n = 0$ $\mathcal{L}_0(T) = \{t_0\}$ where t_0 is the root of T and therefore finite Suppose that $\mathcal{L}_n(T)$ is finite. Note that $\mathcal{L}_{n+1}(T) = \bigcup_{t\in\mathcal{L}_n(T)} \text{Succ}_T(t)$. Since T is finitely branching, $\text{Succ}_T(t)$ is finite for every t and by the induction hypothesis, $\mathcal{L}_n(T)$ is finite. Hence $\mathcal{L}_n(T)$ is a finite union of finite sets which is finite.

(ii) Suppose that $\mathcal{L}_n(T)$ is finite for all $n \geq 0$. Then for every $t \in T$, $\text{Succ}_T(t) \subseteq \mathcal{L}_{ht_T(t)+1}(T)$. By our initial assumption $\mathcal{L}_{ht_T(t)+1}(T)$ is finite, thus $\text{Succ}_T(t)$ is finite.

Problem 2. Use König's Lemma to prove that if $\langle A, \prec\rangle$ is a countable partial order, then there exists a linear ordering $<$ of A which extends $\prec$.

Solution. enumerate $A = \{a_n \mid n \in \mathbb{N}\}$. A node in the n^{th}-level of tree T is any extension R of $\prec \cap\{a_0, ..., a_n\} \times \{a_0, ..., a_n\}$ to a linear ordering of $\{a_0, ..., a_n\}$. For example $\mathcal{L}_0(T) = \{\emptyset\}$ and $\emptyset$ linearly orders $\{a_0\}$. If $a_0 \prec a_1$, then $\mathcal{L}_1(T) = \{\{\langle a_0, a_1\rangle\}\}$ and similarly if $a_1 \prec a_0$. But if a_0, a_1 are incomparable in $\prec$, then we let $\mathcal{L}_1(T) = \{\{\langle a_0, a_1\rangle\}, \{\langle a_1, a_2\rangle\}\}$. The tree T is ordered by inclusion. Clearly, each level of T is non-empty since any finite order can be extended to a linear order. By König's Lemma, let

$b = \{R_n \mid n \in \mathbb{N}\}$ be a branch through T, prove that $R = \bigcup_{n\in\mathbb{N}} R_n$ is a linear ordering of A which extends $\prec$.

Problem 3. In class we proved Ramsey's theorem for countable graphs, that is: If $c : [\mathbb{N}]^2 \to \{0, 1\}$ is any coloring, then there is $H \subseteq \mathbb{N}$ infinite such that $c \restriction [H]^2$ is constant. We called such H a c-monochromatic set.

(i) Prove that for any $r \in \mathbb{N}$ and for every $c : [\mathbb{N}]^2 \to \{0, 1, ..., r\}$ there is an infinite set $H \subseteq \mathbb{N}$ which is c-monochromatic.

(ii) (Optional) Prove that for any $r, s \in \mathbb{N}$ and for every $c : [\mathbb{N}]^s \to \{0, 1, ..., r\}$ there is an infinite set $H \subseteq \mathbb{N}$ which is c-monochromatic.

Solution. The solution is very similar to the one in class. Here is a link to a proof

Problem 4. In this exercise, you will prove the Compactness Theorem from König's Lemma. Given a countable $\Sigma \subseteq \overline{\mathcal{L}}$ which is finitely satisfiable, let us define a tree. First we enumerate $\Sigma = \{\sigma_0, \sigma_1, \sigma_2, ...\}$, and let $\mathcal{L} = \{v_0, v_1, ...\}$ be an enumeration of all the sentence symbols.

(a) Each function $\phi : \{0, ..., n\} \to \{0, 1\}$, can be identified with a functions $\phi^* : \Gamma_n \to \{T, F\}$ where $\Gamma_n = \{v_0, ..., v_n\}$. Describe the identification. No proof required.

Solution. We identify ϕ with the function $\phi^*(v_i) = T$ if and only if $\phi(i) = 1$.

(b) Let $\Sigma_n \subseteq \Sigma$ be the set of all $\sigma \in \Sigma$ which mentions only sentence symbols from Γ_n. Show by induction on the length of σ, that for any

TVA V, the value of $\bar{V}(\sigma)$ depends only on $V \restriction \Gamma_n$, namely if V_1, V_2 are TVA's such that $V_1 \restriction \Gamma_n = V_2 \restriction \Gamma_n$ then $V_1(\sigma) = V_2(\sigma)$.

Solution. If σ is a sentence symbol then $\in \Gamma_n$ and therefore if $V_1 \restriction \Gamma_n = V_2 \restriction \Gamma_n$ then $V_1(\sigma) = V_2(\sigma)$. Suppose that this is true for α, β which mentions only Γ_n, and consider the following cases:

(1) $\bar{V_1}(\neg\alpha) = T$ iff $\bar{V_1}(\alpha) = F$ iff $\bar{V_2}(\alpha) = F$ iff $\bar{V_2}(\neg\alpha) = T$.

(2) $\bar{V_1}(\alpha \wedge \beta) = T$ iff $\bar{V_1}(\alpha) = T$ and $\bar{V_1}(\beta) = T$ iff $\bar{V_2}(\alpha) = T$ and $\bar{V_2}(\beta) = T$ iff $\bar{V_2}(\alpha \wedge \beta) = T$.

The other cases are similar.

(c) Define the tree $T \subseteq T_2$ as follows: at level n, we put all the function ϕ such that some (any) TVA which extends ϕ^* satisfies Σ_n. We order T as usual by end-extension of functions. Prove that if T has an infinite branch then Σ is satisfiable.

Solution. Let $b = \{b_n \mid n \in \mathbb{N}\}$ be an infinite branch. then $b_n \in \mathcal{L}_n(T)$ and $b_n : \{0, ..., n\} \to \{0, 1\}$. Let $\phi = \bigcup_{n \in \mathbb{N}} b_n$. Then ϕ is a function since the order of the tree is just inclusion and for every n, $\phi \restriction \{0, ..., n\} = b_n$. Let us prove that $\phi^* = V$ satisfies Σ. Let $\sigma \in \Sigma$, then there is n such that $\sigma \in \Gamma_n$. By assumption, there is ψ^* satisfying σ such that ψ extends b_n. Since both V and ψ^* agree on Γ (as both ϕ, ψ extend b_n), by the previous item, $V(\sigma) = \Psi^*(\sigma) = T$.

(d) Show that for every n, $\mathcal{L}_n(T) \neq \emptyset$. Namely, prove that for each n, there is $\phi : \{0, ..., n\} \to \{0, 1\}$ such that ϕ^* extends to a TVA which satisfy Σ_n. This proof is done in a few steps:

(a) Prove the existence of ϕ^* in case Σ_n is finite. **Solution.** This is because Σ is finitely satisfiable.

(b) If Σ_n is infinite, enumerate $\Sigma_n = \{\sigma_0, \sigma_1, ...\}$ and for each k prove that there is a TVA ϕ_k such that satisfying $\{\sigma_0, ..., \sigma_k\}$. **Solution.** Agaim, this is because Σ is finitely satisfiable.

(c) Use the pigeonhole principle to find a single $\phi : \{0, ..., n\} \to \{0, 1\}$ such that for infinitely many k's $\phi_k \restriction \{v_0, .., v_n\} = \phi^*$. **Solution.** There are only finitely many possibilities for $\phi_n \restriction \{v_0, ..., v_n\}$ so there must be infinitely many $k's$ such that $\phi_k \restriction \{v_0, ..., v_n\} = \phi^*$.

(d) Prove that $\phi \in \mathcal{L}_n(T)$.

Solution. let $\sigma \in \Sigma_n$, then there is i such that $\sigma = \sigma_i$. Pick $k \geq i$ so that $\phi_k \restriction \{v_0, ..., v_n\} = \phi^*$, then $\phi_k(\sigma_i) = \phi^*(\sigma_i)$ (since they agree on Γ_n) and by the choice of ϕ_k, $\phi^*(\sigma_i) = T$. $\phi_i(\sigma_i)$

(e) Use König's Theorem to show that T has an infinite branch and deduce that Σ is satisfiable. **Solution.** By König's Lemma, there is a branch ϕ and by (c) we are done.