Homework 2-Sols

MATH 461 (due February 9) Feb 2, 2024

Problem 1. Let $C(\mathbb{R})$ be the set of all continuous function $f : \mathbb{R} \to \mathbb{R}$. Prove that

$$C(\mathbb{R}) \leq {}^{\mathbb{Q}}\mathbb{R}$$

[Hint: use that fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ to prove that the restriction function $G : C(\mathbb{R}) \to {}^{\mathbb{Q}}\mathbb{R}$ defined by $G(f) = f \restriction \mathbb{Q}$ is one-to-one.]

solution. Let $G : C(\mathbb{R}) \to {}^{\mathbb{Q}}\mathbb{R}$ defined by $G(f) = f \restriction \mathbb{Q}$, let us prove that it is one-to-one. Suppose that f, g are two continuous functions, such that $f \restriction \mathbb{Q} = g \restriction \mathbb{Q}$. We need to prove $f = g$. Let $x \in \mathbb{R}$, by density of the rationals we can find a sequence $(q_n)_{n=0}^{\infty}$ of rationals, such that $\lim_{n\to\infty} q_n = x$, then for each n, $f(q_n) = g(q_n)$ (since $f \restriction \mathbb{Q} = g \restriction \mathbb{Q}$). By continuity,

$$f(x) = \lim_{n\to\infty} f(q_n) = \lim_{n\to\infty} g(q_n) = g(x)$$

Problem 2. Prove that for every $\alpha < \beta$ real numbers $(\alpha, \beta) \approx (0, 1)$. [Hint: First stretch/shrink $(0, 1)$ to have length $\beta - \alpha$, then shift it by $+c$ as we did in class.]

Solution. The function $f : (\alpha, \beta) \to (0, 1)$ defined by $f(x) = \frac{x-\alpha}{\beta-\alpha}$ is a bijection (check it!)

Problem 3. Show that ${}^{\mathbb{N}}\{0, 1\} \times {}^{\mathbb{N}}\{0, 1\} \approx {}^{\mathbb{N}}\{0, 1\}$. [Hint: see HW1 Problem 5.]

Solution the interleaving function was shown to be a bijection by this exercise and therefore the sets are enumerable.

Problem 4. Prove by a diagonalization argument that $\mathbb{N} \prec {}^{\mathbb{N}}\mathbb{N}_{even}$.

solution First, the function $G : \mathbb{N} \to {}^{\mathbb{N}}\mathbb{N}_{even}$ defined by $G(n)(m) = 2n$ (namely the function G maps the natural number n to the constant function $2n$) is injective. Now suppose towards a contradiction that $F : \mathbb{N} \to {}^{\mathbb{N}}\mathbb{N}_{even}$ is subjective. Define

$$g(n) = F(n)(n) + 2$$

Note that $F(n)(n)$ is even and therefore $F(n)(n) + 2 > F(n)(n)$ is also even. Hence $g : \mathbb{N} \to \mathbb{N}_{even}$ and for every n, $g(n) \neq F(n)(n)$. It follows that $g \neq F(n)$ for every n, which in turn implies that $g \notin Im(F)$, contradicting F being onto.

Problem 5. Prove that $\{X \in P(\mathbb{N}) \mid X \approx \mathbb{N}\} \approx P(\mathbb{N})$. [Hint: Cantor-Bernstein]

Solution: Denote by A the set on the left-hand side. The $A \subseteq P(\mathbb{N})$ and therefore $A \preceq P(\mathbb{N})$. In the other direction, since $\mathbb{N} \approx \mathbb{N}_{even}$ it follows that $P(\mathbb{N}) \approx P(\mathbb{N}_{even})$ so is suffices to find an injection $f : P(\mathbb{N}_{even}) \to A$. Define $f(X) = X \cup \mathbb{N}_{odd}$. Then f is well defined as for every $X \in P(\mathbb{N}_{even})$,

$$\mathbb{N}_{odd} \subseteq f(X) = X \cup \mathbb{N}_{odd} \subseteq \mathbb{N}$$

Hence by CBS theorem, $f(X) \approx \mathbb{N}$. It follows that $f(X) \in A$. To see that f is injective, Suppose that $X_1, X_2 \subseteq \mathbb{N}_{even}$ and $f(X_1) = f(X_2)$. Then

$$\mathbb{N}_{even} \cap f(X_1) = \mathbb{N}_{even} \cap (X_1 \cup \mathbb{N}_{odd}) = \mathbb{N}_{even} \cap (X_2 \cup \mathbb{N}_{odd} = \mathbb{N}_{even} \cap f(X_2)$$

By distributivity of $\cap, \cup$, and since $X_1, X_2 \subseteq \mathbb{N}_{even}$, we have for $i = 1, 2$

$$\mathbb{N}_{even} \cap (X_i \cup \mathbb{N}_{odd}) = (\mathbb{N}_{even} \cap X_i) \cup (\mathbb{N}_{even} \cap \mathbb{N}_{odd}) = X_i \cup \emptyset = X_i$$

Hence $X_1 = X_2$.

Problem 6. Let A be any set. Let us define recursively $A_0 = A$ and $A_{n+1} = P(A_n)$. Define $A_\infty = \bigcup_{n\in\mathbb{N}} A_n$. Prove that for every set A and any $n \in \mathbb{N}$, $A_n \prec A_\omega$.

Solution. By inclusion, for every $n \in \mathbb{N}$, $A_n \preceq A_\infty$. Suppose toward a contradiction that there is n such that $A_n \approx A_\infty$.Then $A_n \preceq A_{n+1} \preceq A_\infty \approx A_n$ and therefore by CSB Theorem $A_{n+1} \approx A_n$. However, $A_{n+1} = P(A_n)$, this is a contradiction to Cantor's theorem that for every set A, $A \prec P(A)$.