Homework 10

MATH 461 (due April 19) April 12, 2024

Problem 1. Prove that a set of Γ WFF is consistent iff every finite subset $\Gamma_0 \subseteq \Gamma$ is consistent.

Solution. If Γ is consistent i.e. does not prove a contradiction. Let $\Gamma_0 \subseteq \Gamma$, since any Γ_0-deduction is also a Γ-deduction, Γ_0 cannot prove a contradiction and therefore Γ_0 is consistent. in the other direction, suppose that Γ is inconsistent and let β be such that $\Gamma \vdash \beta$ and $\Gamma \vdash \neg\beta$. Then there are deductions $\langle \alpha_1, ..., \alpha_n = \beta \rangle$ and $\langle \beta_1, ..., \beta_n = \neg\beta \rangle$ from Γ. These deductions only involve finitely many formulas from Γ and therefore there is $\Gamma_0 \subseteq \Gamma$ finite such that $\langle \alpha_1, ..., \alpha_n = \beta \rangle$ and $\langle \beta_1, ..., \beta_n = \neg\beta \rangle$ are Γ_0-deductions, namely $\Gamma_0 \Vdash \beta$ and $\Gamma_0 \Vdash \neg\beta$.

Problem 2. Using Problem 1, conclude the compactness theorem from the completeness theorem [Hint: Use the equivalent formulation we presented in class for the completeness theorem].

Solution. Proven in class.

Problem 3. (i) Prove by induction on the length of the WFF ϕ that if y does not occur in ϕ, then x is substitutable for y in ϕ^x_y and $(\phi^x_y)^y_x = \phi$.
Solution. First note this for any term without the variable y: If $t = z$ then $z \neq y$ (since y does note occur in t). If $z \neq x$, then $(z^x_y)^y_x = (z)^y_x = z$ and if $z = x$ then $(z^x_y)^y_x = y^y_x = x = z$. This is similar for constants (as there are no variables in constants) and for terms $t = f(t_1, ..., t_n)$ we have that y does not occur in t iff y does not occur in $t_1, ..., t_n$ and in particular by the induction hypothesis

$$(f((t_1, ..., t_n))^x_y)^y_x = f(((t_1)^x_y)^y_x, ..., (t_n))^x_y)^y_x) = f(t_1, ..., t_n)$$

For atomic formulas, $t_1 = t_2$ without y, y does not appear in t_1, t_2 and therefore $((t_1 = t_2)^x_y)^y_x$ is the same as $((t_1)^x_y)^y_x = ((t_2)^x_y)^y_x$ which by the previous part is $t_1 = t_2$. For $(P(t_1, ..., t_n))^x_y)^y_x$ this is similar. The general case is also similar.

(ii) Give an example of a WFF ϕ such that $(\phi^x_y)^y_x \neq \phi$.

Solution take ϕ to be $y = x$.