MidTerm I- Set Theory fall 2023

MATH 361 (Instructor: Tom Benhamou) October 06

Instructions

The midterm duration is 1 hour and 20 min, and consists of 4 problems, each worth 26 points (The maximal grade is 100). The answers to the problems should be written in the designated areas.

Problems

Problem 1. Prove that if $A \cup B = A \cup C$ and $A \cap B = A \cap C$ then $B = C$.

Solution: Assume $A \cup B = A \cup C$ and $A \cap B = A \cap C$. Le us rove by double inclusion that $B = C$.

$\subseteq$: Let $b \in B$ and let us split into cases:

(a) If $b \in A$, then $b \in A \cap B$ and since $A \cap B = A \cap C$, $b \in A \cap C$. In particular $b \in C$.

(b) If $b \notin A$ then since $b \in B$ it follows that $b \in A \cup B$. Since $A \cup B = A \cup C$, $b \in A \cup C$. Since we assumed that $b \in A$ it follows that $b \in C$.

In any case $b \in C$.

$\supseteq$ The other direction is symmetric.

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Problem 2. Suppose that $f : A \to B$, $g : B \to C$ are any functions such that f is onto and g is **not** injective. Prove that $g \circ f$ is **not** injective.

Solution: Suppose that $f : A \to B$, $g : B \to C$ are any functions such that f is onto and g is **not** injective. This means that there are $b_1 \neq b_2$ in B such that $g(b_1) = g(b_2)$. Since f is onto, there are $a_1, a_2 \in A$ such that $f(a_1) = b_1$ and $f(a_2) = b_2$. Since f is univalent and $b_1 \neq b_2$, it follows that $a_1 \neq a_2$. It follows that $g \circ f(a_1) = g(f(a_1)) = g(b_1) = g(b_2) = g(f(a_2)) = g \circ f(a_2)$, hence $g \circ f$ is not injective.

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Problem 3. For a function $f : A \to B$ and $C \subseteq A$ define the *pointwise image of* C *by* f as

$$f''C = \{f(c) \mid c \in C\}$$

Prove that if $f : A \to B$ is injection and $C \subseteq A$, then

$$(f''A) \setminus (f''C) = f''[A \setminus C].$$

Solution:

$\subseteq$: Let $b \in f''A \setminus f''C$. Since $b \in f''A$, there is $a \in A$ such that $b = f(a)$. Since $b \notin f''C$, $a \notin C$. It follows that $a \in A \setminus C$. We conclude that $b = f(a) \in f''[A \setminus C]$.

$\supseteq$: For the other direction, let $x \in f''[A \setminus C]$. Then there is $a \in A \setminus C$ such that $f(a) = x$. By the definition of difference, we would like to prove that $x \in f''A$ and $x \notin f''C$. Since $a \in A$, it follows that $x = f(a) \in f''A$. Suppose towards a contradiction that there is $c \in C$ such that $f(c) = x$. Then $f(c) = f(a)$. Since f is injective, $c = a$. However $c \in C$ and $a \notin C$, contradiction. Hence $x \in f''C$.

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Problem 4. On the set ${}^{\mathbb{R}}\mathbb{R}$ we define the relation:

$$f \sim g \text{ if and only if } \exists \epsilon > 0,\ f \restriction (0, \epsilon) = g \restriction (0, \epsilon).$$

Prove that $\sim$ is an equivalence relation on ${}^{\mathbb{R}}\mathbb{R}$.

Solution: Let us prove that $\sim$ is reflexive, symmetrice and transitive.

1. Reflexive: Let $f : \mathbb{R} \to \mathbb{R}$. Then for $\epsilon = 1$ we have that $f \restriction (0, 1) = f \restriction (0, 1)$ and therefore there exists $\epsilon > 0$ such that $f \restriction (0, \epsilon) = f \restriction (0, \epsilon)$ which implies that $f \sim f$.

2. Symmetric: Suppose that $f \sim g$, then there is $\epsilon > 0$ such that $f \restriction (0, \epsilon) = g \restriction (0, \epsilon)$. Since equality is symmetric, $g \restriction (0, \epsilon) = f \restriction (0, \epsilon)$ and therfore $g \sim f$.

3. Suppose that $f \sim g$ and $g \sim h$. Then there are $\epsilon_1, \epsilon_2 > 0$ such that $f \restriction (0, \epsilon_1) = g \restriction (0, \epsilon_1)$ and $g \restriction (0, \epsilon_2) = h \restriction (0, \epsilon_2)$. Take $\epsilon_3 = \min(\epsilon_1, \epsilon_2)$. Then $(0, \epsilon_3) \subseteq (0, \epsilon_1)$ and therefore $f \restriction (0, \epsilon_3) = g \restriction (0, \epsilon_3)$. Also, $(0, \epsilon_3) \subseteq (0, \epsilon_2)$ and therefore $g \restriction (0, \epsilon_3) = h \restriction (0, \epsilon_3)$. We conclude that $f \restriction (0, \epsilon_3) = h \restriction (0, \epsilon_3)$ and so $f \sim h$.