Homework 8-Solutions

MATH 361 (due November 20) November 11, 2022

Problem 1. Prove that $\{X \in P(\mathbb{N}) \mid X \text{ is infinite}\} \approx P(\mathbb{N})$

Solution. See HW7 Problem 4.

Problem 2. Determine the cardinality ($\aleph_0, 2^{\aleph_0}, 2^{2^{\aleph_0}}, \ldots$) of the following sets (submit only 3 of the items):

We give crushed solutions, with the main ideas of all the function. Of course you should have more details in your solutions.

(1) $A = \{f \in {}^{\mathbb{N}}\{0,1\} \mid \forall n \in \mathbb{N}_{even}, f(n) = 1\}$.

Solution. $|A| = 2^{\aleph_0}$. Indeed, $A \subseteq {}^{\mathbb{N}}\{0,1\}$ and therefore $A \preceq {}^{\mathbb{N}}\{0,1\}$ and the function $F : {}^{\mathbb{N}_{odd}}\{0,1\} \to A$ defined by $F(g)(n) = \begin{cases} f(n) & n \in \mathbb{N}_{odd} \\ 1 & n \in \mathbb{N}_{even} \end{cases}$ is injective (check that it is injective and well-defined!). Since ${}^{\mathbb{N}_{odd}}\{0,1\} \approx {}^{\mathbb{N}}\{0,1\}$ we conclude that

$$^{\mathbb{N}}\{0,1\} \preceq A$$

By CSB, $A \approx \{0,1\}^{\mathbb{N}}$, so $|A| = 2^{\aleph_0}$.

(2) $B = \{X \in P(\mathbb{N}) \mid X \text{ contains no consecutive numbers}\}$.

Solution. $P(\mathbb{N}_{even}) \subseteq B \subseteq P(\mathbb{N})$ hence $|B| = 2^{\aleph_0}$.

(3) The set of all arithmetic progressions of integers. [Recall: an arithmetic progression of integers is a sequence $(a_n)_{n=0}^{\infty}$ such that for some d, for any n, difference $a_{n+1} - a_n = d$.]

Solution. Let AP by the set of arithmetic progressions. Any arithmetic progression is uniquely determined (namely, there is a one-to-one function) by (a_0, d). Hence

$$AP \leq \mathbb{N} \times \mathbb{N}$$

. Show that AP is infinite and deduce that $|AP| = \aleph_0$.

(4) The set of all circles in the plain.[Given a point $p = \langle x_0, y_0 \rangle \in \mathbb{R}^2$ ("the center") and $r \in (0, \infty)$ ("the radious"), the circle $C = C(p, r) = \{\langle x, y \rangle \in \mathbb{R}^2 \mid (x - x_0)^2 + (y - y_0)^2 = r^2\}$. A]

Solution. A circle is uniquely determined by the center and the radius, hence there is a bijection with $\mathbb{R} \times \mathbb{R}$. It follows that there are $2^{\aleph_0}$-many such circles.

(5) The set of all circles C in $\mathbb{R}^2$ which intersect the x-axis at two points $\langle 0, q_1 \rangle, \langle 0, q_2 \rangle$, where $q_1, q_2 \in \mathbb{Q}$.

Solution. Two points on the circle determines at most two circles (solve the equations!) hence the set is a countable union (over $\langle q_1, q_2 \rangle \in \mathbb{Q}$) of sets of size at most 2 hence countable. It follows that $|C| = \aleph_0$.

Problem 3. A straight line in the plain is a set of the following forms:

- $L_c = \{c\} \times \mathbb{R}$ for some $c \in \mathbb{R}$ (lines which are parallel to the y-axis).
- $L_{a,b} = \{\langle x, y \rangle \in \mathbb{R} \mid y = ax + b\}$ for some $a, b \in \mathbb{R}$. (lines which are not parallel to the y-axis)

Answer the following questions:

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1. What is the cardinality of the set of all lines in the plain?

   Solution. We need to compute the cardinality of $\mathcal{L} = \{L_c \mid c \in \mathbb{R}\} \cup \{L_{a,b} \mid a, b \in \mathbb{R}\}$. Clearly there is an injection from $\mathbb{R}$ to $\mathcal{L}$ (for example $f(r) = L_r$). So $\mathbb{R} \leq \mathcal{L}$. For the other direction, we can define an onto function from $\mathbb{R}^3$ to $\mathcal{L}$ by

   $$g(\langle a, b, c\rangle) = \begin{cases} L_{a,b} & c = 0 \\ L_a & c \neq 0 \end{cases}$$

   Hence $|\mathcal{L}| = 2^{\aleph_0}$

2. Prove that there is a line of the form $L_{a,b}$ which contains no rational point, namely $L \cap \mathbb{Q} \times \mathbb{Q} = \emptyset$.

   Solution. $L_{\sqrt{2},0}$ is such a line, since if $\langle x, y\rangle \in L_{\sqrt{2},0}$ then $y = \sqrt{2}x$ and if x is rational then y cannot be rational (otherwise $\sqrt{2}$ would have been rational).

3. **(A typo in the original formulation of the problem)** Prove that every line of the form $L_{a,b}$ for $a > 0$ contains an irrational point, namely $L \cap (\mathbb{R} \setminus \mathbb{Q}) \times (\mathbb{R} \setminus \mathbb{Q}) \neq \emptyset$.

   Solution. Just otherwise, for every $\langle x, y\rangle \in L_{a,b}$, wither $x \in \mathbb{Q}$ or $y \in \mathbb{Q}$, So $L \subseteq A \cup B$ where $A = \{\langle q, aq + b\rangle \mid q \in \mathbb{Q}\}$ and $B = \{\langle \frac{q-b}{a}, a\rangle \mid q \in \mathbb{Q}\}$. Both A, B are clearly countable and therefore $A \cup B$ is countable. It follows that $L_{a,b}$ is countable, contradiction.

Problem 4. A function $f : \mathbb{N} \to \mathbb{N}$ is increasingly monotone, if for every n, $f(n) < f(n+1)$. Prove that the set A of all increasingly monotone functions

$f : \mathbb{N} \to \mathbb{N}$ has cardinality $2^{\aleph_0}$. [Hint: CSB. One direction is easy. For the other, given a function $f : \mathbb{N} \to \mathbb{N}_+$, define $F(f)(n) = \sum_{k=0}^{n} f(k)$.]

Solution. Let M be the set of monotone functions, then $M \subseteq {}^{\mathbb{N}}\mathbb{N}$ which we saw in class has cardinality $2^{\aleph_0}$. For the other direction, for any function $f : \mathbb{N} \to \mathbb{N}_+$, $F(f)(n) = \sum_{k=0}^{n} f(k)$. We claim that $F : {}^{\mathbb{N}}\mathbb{N}_+ \to M$ is injective and clearly, ${}^{\mathbb{N}}\mathbb{N}_+$ has cardinality $2^{\aleph_0}$ in which case we will be done. To see this, first note that

$$F(f)(n+1) = \sum_{k=0}^{n+1} f(k) = \sum_{k=0}^{n} f(k) + f(n+1) > \sum_{k=0}^{n} f(k) = F(f)(n)$$

Hence $F(f) \in M$. To see it is one-to-one, suppose that $F(f) = F(g)$. We prove by induction that for every n, $f(n) = g(n)$. Indeed, $f(0) = F(f)(0) = F(g)(0) = g(0)$. Suppose this holds up to n, and let us prove that $f(n+1) = g(n+1)$.

$$F(g)(n+1) = \sum_{k=0}^{n+1} g(k) = F(f)(n+1) = \sum_{k=0}^{n+1} f(k)$$

Hence

$$(*) \quad \sum_{k=0}^{n} g(k) + g(n+1) = \sum_{k=0}^{n} f(k) + f(n+1)$$

By the induction hypothesis $\sum_{k=0}^{n} g(k) = \sum_{k=0}^{n} f(k)$, so we get that from $(*)$ that $f(n+1) = g(n+1)$.

Problem 5. Prove that $\aleph_0^{(2^{\aleph_0})} = 2^{(2^{\aleph_0})}$.

Solution.

$$2^{(2^{\aleph_0})} \leq^{\text{mon}} \aleph_0^{(2^{\aleph_0})} \leq (2^{\aleph_0})^{(2^{\aleph_0})} = 2^{(\aleph_0 \cdot 2^{\aleph_0})} = 2^{(2^{\aleph_0})}$$

By CBS we conclude the equiality.

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Problem 6. Prove that $\kappa^{\lambda+\sigma} = \kappa^{\lambda} \cdot \kappa^{\sigma}$.

Solution. I will just give the function. Suppose that $|A| = \kappa$, $|B| = \lambda$ and $|C| = \sigma$, such that $B \cap C = \emptyset$. Define $F : {}^{B\cup C}A \to {}^{B}A \times {}^{C}A$ by

$$F(h) = \langle h \restriction B, h \restriction C \rangle$$

1 Additional problems- preparation for midterm II

Problem 7. Compute the cardinality of the set of all function $f : \mathbb{N} \to \{0,1\}$ with no consecutive zeros. Namely, there is no $n \in \mathbb{N}$ such that $f(n) = f(n+1) = 0$.

Problem 8. Consider the relation E om ${}^{\mathbb{N}}\mathbb{N}$ by fEg if and only if for every $n \geq 100$, $f(n) = g(n)$.

1. Prove that E is an equivalence relation.
2. Compute the cardinality of ${}^{\mathbb{N}}\mathbb{N}/E$.

Problem 9. Let $\leq_A, \leq_B$ be two weak linear orderings of A, B (resp.), where A, B are disjoint. We define $\leq_A + \leq_B$ which we abbreviate by $\leq_+$ on AB as follows:

$$x \leq_+ y \leftrightarrow (x, y \in A \wedge x \leq_A y) \vee (x, y \in B \wedge x \leq_B y) \vee (x \in A \wedge y \in B)$$

1. Prove that $\leq_+$ is a linear ordering of $A \cup B$.
2. Let $\mathbb{N}^* = \{0\} \times \mathbb{N}$ and define $\leq^*$ on $\mathbb{N}^*$ by $\langle 0, n \rangle \leq^* \langle 0, m \rangle$ if and only if $m \leq n$. Prove that $\leq^*$ is a linear ordering of $\mathbb{N}^*$.

3. Prove that $\langle \mathbb{N}^* \cup \mathbb{N}, \leq^* + \leq \rangle \simeq \langle \mathbb{Z}, \leq \rangle$.

Problem 10. Define recursively $A_0 = \emptyset$ and $A_{n+1} = P(A_n)$. Prove by induction that for every n, $A_n \subseteq A_{n+1}$.

Problem 11. Prove that the intersection of finitely many Dedekind cuts is a Dedekind cut.

Problem 12. Prove that if $x \in \mathbb{R}$ and $y \in \mathbb{R}$ is positive (namely $0 < y$), then $x < x + y$.