Homework 7-Solutions

MATH 361 (due November 11) November 4, 2023

Problem 1. Prove that if E is an equivalence relation on A and let A' be a system of representatives.

1. Prove that $A/E \approx A'$.

   Solution. Let us define $f : A' \to A/E$ by $f(a) = [a]_E$. The f is one-to-one as if $a \neq a'$ are two representatives then $a \not{E} a'$ which implies that
$$f(a) = [a]_E \neq [a']_E = f(a')$$
   To see that f is onto A/E, let $[b]_E \in A/E$. Since A' is a system of representatives, there is $a' \in A$ such that $a'Eb$ and therefore $f(a') = [a']_E = [b]_E$.

2. Conclude that $A/E \preceq A$. [Remark: first prove it using the previous item (one line proof). Then try to prove it directly by finding an onto function from A to A/E].

   Solution. By (1), $A/E \approx A'$ and since $A' \subseteq A$, $A/E \approx A' \preceq A$. We can prove it directly by defining $g : A \to A/E$ by $g(a) = [a]_E$. This function is onto and therefore by the theorem in class $A/E \preceq A$.

Problem 2. Let $A = {}^{\mathbb{N}}\mathbb{N}$, and consider the equivalence relation $R = \{\langle f, g\rangle \in (\mathbb{N}^{\mathbb{N}})^2 \mid f(0) = g(0)\}$ in A (no need to prove that). Prove that $A/R \approx \mathbb{N}$. [Hint: Use problem 1]

Solution. Consider the function $f_n(m) = \begin{cases} n & m = 0 \\ 0 & m > 0 \end{cases}$. Check that $\{f_n \mid n \in \mathbb{N}\}$ is a system of representatives. Clearly, $n \neq m$ implies that $f_n \neq f_m$ as therefore this set is infinitely countable. Hence by the previous problem

$$^{\mathbb{N}}\mathbb{N}/R \approx \{f_n \mid n \in \mathbb{N}\} \approx \mathbb{N}$$

Problem 3. Prove by a diagonalization argument that $\mathbb{N} \prec {}^{\mathbb{N}}\mathbb{N}_{even}$.

Solution. Find. the function $G : \mathbb{N} \to {}^{\mathbb{N}}\mathbb{N}_{even}$ defined by $G(n)(m) = 2n$ (namely the function G maps the natural number n to the constant function $2n$) is injective. Now suppose towards a contradiction that $F : \mathbb{N} \to {}^{\mathbb{N}}\mathbb{N}_{even}$ is subjective. Define

$$g(n) = F(n)(n) + 2$$

Note that $F(n)(n)$ is even and therefore $F(n)(n) + 2 > F(n)(n)$ is also even. Hence $g : \mathbb{N} \to \mathbb{N}_{even}$ and for every n, $g(n) \neq F(n)(n)$. It follows that $g \neq F(n)$ for every n, which in turn implies that $g \notin Im(F)$, contradicting F being onto.

Problem 4. Prove that the set of all matrices (of any dimension) with rational entries is countable. [Hint: a countable union of countable sets]

Solution. For every $\langle n, m\rangle \in \mathbb{N}_+ \times \mathbb{N}_+$ let $M_{n,m}[\mathbb{Q}]$ denote the set of all matrices of dimantion $n \times m$. You can check that $f_{n,m} : M_{n,m}[\mathbb{Q}] \to \mathbb{Q}^{n\cdot m}$ defined by

$$f_{n,m}(A) = \langle (A)_{1,1}, ..., (A)_{1,n}, (A)_{2,1}, ..., (A)_{2,n}, ..., (A)_{m,1}, ..., (A)_{m,n}\rangle$$

is a bijection. and therefore $M_{n,m}[\mathbb{Q}] \approx \mathbb{Q}^{m\cdot n} \approx \mathbb{N}$. Now the set

$$M = \bigcup_{\langle n,m\rangle \in \mathbb{N}_+ \times \mathbb{N}_+} M_{n,m}[\mathbb{Q}]$$

is a countable union of countable sets, hence countable.

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Problem 5. Prove that $\{X \in P(\mathbb{N}) \mid X \approx \mathbb{N}\} \approx P(\mathbb{N})$. [Hint: Cantor-Schroeder-Bernstein]

Solution. Denote by A the set on the right hand side. The $A \subseteq P(\mathbb{N})$ and therefore $A \preceq P(\mathbb{N})$. In the other direction, since $\mathbb{N} \approx \mathbb{N}_{even}$ it follows that $P(\mathbb{N}) \approx P(\mathbb{N}_{even})$ so is suffices to find an injection $f : P(\mathbb{N}_{even}) \to A$. Define $f(X) = X \cup \mathbb{N}_{odd}$. Then f is well defined as for every $X \in P(\mathbb{N}_{even})$,

$$\mathbb{N}_{odd} \subseteq f(X) = X \cup \mathbb{N}_{odd} \subseteq \mathbb{N}$$

Hence by CBS theorem, $f(X) \approx \mathbb{N}$. It follows that $f(X) \in A$. To see that f is injective, Suppose that $X_1, X_2 \subseteq \mathbb{N}_{even}$ and $f(X_1) = f(X_2)$. Then

$$\mathbb{N}_{even} \cap f(X_1) = \mathbb{N}_{even} \cap (X_1 \cup \mathbb{N}_{odd}) = \mathbb{N}_{even} \cap (X_2 \cup \mathbb{N}_{odd} = \mathbb{N}_{even} \cap f(X_2)$$

By distributivity of $\cap, \cup$, and since $X_1, X_2 \subseteq \mathbb{N}_{even}$, we have for $i = 1, 2$

$$\mathbb{N}_{even} \cap (X_i \cup \mathbb{N}_{odd}) = (\mathbb{N}_{even} \cap X_i) \cup (\mathbb{N}_{even} \cap \mathbb{N}_{odd}) = X_i \cup \emptyset = X_i$$

Hence $X_1 = X_2$.

1 Additional Problems

Problem 6. A function $f : A \to B$ is called countable-to-one if every $b \in B$ has at most countably many preimages. Namely, if for every $b \in B$, the following set is countable:

$$\{a \in A \mid f(a) = b\}$$

1. Give an example of a function which is countable-to-one but not one-to-one.

2. Suppose that A is a set such that there exists a countable-to-one function $f : A \to \mathbb{Q}$. Prove that A is countable. [Hint: countable union of countable sets is countable]

Problem 7. On ${}^{\mathbb{N}}\{0,1\}$, define the equivalence relation E by fEg if and only if there is N such that for every $n \geq N$, $f(n) = g(n)$.

Prove that ${}^{\mathbb{N}}\{0,1\}/E \approx {}^{\mathbb{N}}\{0,1\}$. [Guidence: In order to prove that ${}^{\mathbb{N}}\{0,1\} \preceq {}^{\mathbb{N}}\{0,1\}/E$, decompose $\mathbb{N}$ to infinitely many infinite disjoint sets $\mathbb{N} = \biguplus_{n\in\mathbb{N}} A_n$. Try to use such a decomposition to define a function $F : {}^{\mathbb{N}}\{0,1\} \to {}^{\mathbb{N}}\{0,1\}$ which duplicates each value of the in input value f (i.e. duplicates the values $f(n)$) infinitely many times]