Homework 6

MATH 361 (due November 4) October 28 2022

Problem 1. Prove that for any $r, s \in \mathbb{R}$, $r \cdot s \in \mathbb{R}$. (You can problem 6 from HW5).

Problem 2. For each of the following statements provide an appropriate function (no need to prove that your functions satisfy the required properties):

1. $\mathbb{R} \approx \mathbb{R} \setminus \{0\}$.
2. $\mathbb{Z} \approx \mathbb{N}_{even} \setminus \{0, \dots, 2023\}$.
3. $\mathbb{N} \times \mathbb{N} \preceq {}^{\mathbb{N}}\{0, 1\}$
4. $\left\{f \in {}^{\mathbb{R}}\mathbb{R} \mid \exists i \in \{0, 1\}, \ \forall x \in \mathbb{R} \setminus \mathbb{Q}, \ f(x) = i\right\} \approx \{0, 1\} \times {}^{\mathbb{Q}}\mathbb{R}$.

Problem 3. Prove that

$$\left\{X \in P(\mathbb{N}) \mid \mathbb{N}_{even} \subseteq X\right\} \approx P(\mathbb{N})$$

[Hint: First find a function from $P(\mathbb{N})$ to $P(\mathbb{N}_{odd})$]

Problem 4. Let $C(\mathbb{R})$ be the set of all continuous function $f : \mathbb{R} \to \mathbb{R}$. Prove that

$$C(\mathbb{R}) \preceq {}^{\mathbb{Q}}\mathbb{R}$$

[Hint: use that fact that $\mathbb{Q}$ is dense in $\mathbb{R}$ to prove that the restriction function $G : C(\mathbb{R}) \to {}^{\mathbb{Q}}\mathbb{R}$ defined by $G(f) = f \restriction \mathbb{Q}$ is one-to-one.]

Problem 5. Prove that if $A \approx B$ and $C \approx D$ then $A \times C \approx B \times D$.

Problem 6. Prove that for every $\alpha < \beta$ real numbers $(\alpha, \beta) \approx (0, 1)$. [Hint: First stretch/shrink $(0, 1)$ to have length $\beta - \alpha$, then shift it by $+c$ as we did in class.]

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Additional problems

Problem 7. Show that $x \cdot (y+z) = x \cdot y + x \cdot x$ for every $x, y, z \in \mathbb{R}$.

Problem 8. Show that for every $n > 0$, $\mathbb{N}^n \approx \mathbb{N}$. [Hint: Induction. you can $\mathbb{N} \times \mathbb{N} \approx \mathbb{N}$.]

Problem 9. Show that ${}^{\mathbb{N}}\{0,1\} \times {}^{\mathbb{N}}\{0,1\} \approx {}^{\mathbb{N}}\{0,1\}$. [Hint: see HW2 problem 5.]